
This entire document is based on the coursework during the semester. The problems are based Homework, Midterm and Qualifying exams for the course. This document is strictly meant to help students learn, recap and prepare for better understanding of how to approach problem solving in Complex Analysis.

Note*: This document is regularly updated to include corrections and new facts.

Book Recommendation: Complex Analysis by Lars Ahlfors.

Theory:

- The set of all complex number is denoted by $\mathbb{C}$.
- Any complex number z can be expressed as $z = x + iy$, where $x, y \in \mathbb{R}$ and $i = \sqrt{-1}$.
- The conjugate of z is $\overline{z} = x - iy$.
- $|z|^2 = x^2 + y^2 = z\overline{z}$
- $z + \overline{z} = 2\,\mathrm{Re}(z)$ and $z - \overline{z} = 2i\,\mathrm{Im}(z)$
- $e^z = e^{x+iy} = e^x(\cos y + i \sin y)$
- $e^{in\theta} = (\cos\theta + i\sin\theta)^n = (\cos n\theta + i \sin n\theta)$
- The n-th root of z are the solutions of the equation $\lambda^n = z$ which are $z^{1/n} = r^{1/n}e^{(i\theta+2k\pi)/n}$ where $z = re^{i\theta}$ and $k = 0, 1, 2, \cdots, n-1$
- $\log z = \mathrm{Log}\,|z| + i \arg(z)$ where $\arg(z) = \mathrm{Arg}(z) + 2k\pi$ where $k \in \mathbb{Z}$
- $z^\omega = e^{\log(z^\omega)} = e^{\omega \log z}$
- $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$
- $\cos z = \dfrac{e^{iz} + e^{-iz}}{2}$
- If $f(z)$ is analytic at z_0 and $f'(z_0) \neq 0$ then there is an open disk D centered at z_0 such that $f(z)$ is one-to-one in D.

- f is said to be **conformal at** z_0 if the angles at z_0 are preserved under the mapping f.
- A **conformal map** on a domain D is a non-constant analytic function f on D such that $f'(z) \neq 0$ for each $z \in D$.
- **Open Mapping Theorem:** If $f(z)$ is non-constant and analytic in a domain D, then its range $f(D)$ is an open set in $\mathbb{C}$.
- A Möbius (linear fractional) transformation is any function of the form

$$f(z) = \frac{az+b}{cz+d}$$

  with the restriction $ad - bc \neq 0$.
- Let f be a Möbius transformation. Then
  (i) f can be expressed as the composition of a finite sequence of translations, magnifications, rotations and inversions.
  (ii) f maps the extended complex plane $\widehat{\mathbb{C}}$ bijectively to itself.
  (iii) f maps the class of circles and lines to itself. (If a line or a circle passes through the pole $z = -\frac{d}{c}$ of the Möbius transformation, it is mapped to a straight line (an unbounded figure). A line or circle that avoids the pole is mapped to a circle.)
  (iv) f is conformal at every point except its pole $z = -\frac{d}{c}$.
- Let $z_1, z_2, z_3, z_4 \in \widehat{\mathbb{C}}$ be distinct points. The cross ratio

$$(z, z_2, z_3, z_4) = \frac{(z-z_3)(z_2-z_4)}{(z-z_4)(z_2-z_3)}$$

  defines the Möbius transformation f such that $f(z_2) = 1,\ f(z_3) = 0,\ f(z_4) = \infty$.
  - Case 1: If $z_2, z_3, z_4 \in \mathbb{C}$ are distinct excluding ∞, then $f(z) = \frac{(z-z_3)(z_2-z_4)}{(z-z_4)(z_2-z_3)}$
  - Case 2: If $z_2 = \infty$, then $f(z) = \frac{z-z_3}{z-z_4}$.
  - Case 3: If $z_3 = \infty$, then $f(z) = \frac{z_2-z_4}{z-z_4}$.
  - Case 4: If $z_4 = \infty$, then $f(z) = \frac{z-z_3}{z_2-z_3}$.

- The cross ratio is unique.
- We can find the Möbius transformation carrying z_1, z_2, z_3 to w_1, w_2, w_3 by solving the equation $(w, w_1, w_2, w_3) = (z, z_1, z_2, z_3)$, i.e. equality of the corresponding cross-ratios.
- z and z^* are symmetric with respect to a circle or a straight line C through three distinct points z_1, z_2, z_3 if $(z, z_1, z_2, z_3) = (z^*, z_1, z_2, z_3)$.
- Let $C = \{z : |z - a| = R\}$, $a \in \mathbb{C}$, $R > 0$. Let z_1, z_2, z_3 be three distinct points on the circle C. Let α be a point inside the circle C. Then its symmetric point is $\alpha^* = a + \dfrac{R^2}{\overline{\alpha - a}}$.
  Also, $\arg(\alpha^* - a) = \arg(\alpha - a)$ and $|\alpha^* - a| = \dfrac{R^2}{|\alpha - a|}$.
- **Symmetry Principle:** For a Möbius transformation T that maps a circle (or a straight line) C_1 to a circle (or a straight line) $C_2 = T(C_1)$, it maps any pair of symmetric points z and z^* with respect to C_1 to a pair of symmetric points $w = f(z)$ and $w^* = f(z^*)$ with respect to C_2.
- All Möbius transformations (in fact, all one-to-one analytic mappings) of the unit disk $|z| < 1$ onto itself $|w| < 1$ are of the form

$$f(z) = e^{i\theta}\,\frac{z - \alpha}{1 - \overline{\alpha}\,z}, \qquad |\alpha| < 1,\ \theta \in \mathbb{R}$$

- Property of the Möbius transformation

$$f(z) = \frac{z - z_0}{\overline{z_0}\,z - 1}, \qquad \text{where } |z_0| < 1.$$

  (i) The inverse of f is $f^{-1}(w) = \dfrac{w - z_0}{\overline{z_0}\,w - 1}$

  (ii) $f^{-1} = f$ and $f^2 =$ Identity map.

  (iii) $|f(z)| = 1$ for all $|z| = 1$.

  (iv) f maps $\mathbb{D}$ conformally onto itself and sends $z_0 \mapsto 0$.

  (v) $f'(z_0) = \dfrac{1}{1 - |z_0|^2}$.

- **Assumption on contour:** $\gamma : z = z(t), a \le t \le b$ is a piecewise differentiable curve with $z'(t) \neq 0$ for all $a \le t \le b$.
  Loop = closed contour.

- (i) $\int_\gamma f(z)\,dz := \int_a^b f(z(t))\,z'(t)\,dt$.

  (ii) If $f(z(t)) = u(t) + i\,v(t)$ then $\displaystyle\int_a^b (u(t) + iv(t))\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt$

  (iii) $\int_\gamma |f(z)|\,|dz| = \int_a^b |f(z(t))|\,|z'(t)|\,dt$.

  (iv) $\int_\gamma f(z)\,d\overline{z} := \overline{\int_\gamma \overline{f(z)}\,dz}$.

- Let C_r be the circle $|z - z_0| = r$ traversed once in the counterclockwise direction. Then for integer n,

$$\int_{C_r} (z - z_0)^n\,dz = \begin{cases} 0 & , n \neq -1, \\ 2\pi i & , n = -1. \end{cases}$$

- If f is continuous on the contour γ and $|f(z)| \leq M$ for all $z \in \gamma$, then $\left|\displaystyle\int_\gamma f(z)\,dz\right| \leq M\,l(\gamma)$ where $l(\gamma)$ denotes the length of γ.

- **Fundamental Theorem of Calculus for Contour Integral:**
  Suppose the function $f(z)$ is continuous in a domain D and has an anti-derivative $F(z)$ throughout D, i.e., $\dfrac{dF}{dz}(z) = f(z)$ for each $z \in D$. Then for any contour γ lying in D, with initial point z_0 and terminal point z_1, we have

$$\int_\gamma f(z)\,dz = F(z_1) - F(z_0)$$

- **Deformation Invariance Theorem:** Let f be analytic in a domain D containing the loops γ_0 and γ_1. If these loops can be continuously deformed into one another in D, then

$$\int_{\gamma_0} f(z)\,dz = \int_{\gamma_1} f(z)\,dz$$

- **Cauchy Integral Theorem:** If f is analytic in a simply connected domain D and γ is any loop in D, then

$$\oint_\gamma f(z)\,dz = 0$$

- In a simply connected domain, an analytic function has an antiderivative; hence its contour integrals are independent of path, and its loop integrals vanish.

- **Cauchy Integral Formula:** If f is analytic inside and on the simply closed positively oriented loop γ and if z_0 is any point inside γ, then

$$f(z_0) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z - z_0}\, dz$$

and, for $n = 1, 2, 3, \ldots,$

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_\gamma \frac{f(z)}{(z - z_0)^{n+1}}\, dz$$

- **Morera's Theorem:** If f is continuous in a domain D and if $\oint_\gamma f(z)\, dz = 0$ for every loop γ in D, then f is analytic in D.

- **Cauchy's Estimate:** Let f be analytic inside and on a circle $C = \{z : |z - z_0| = r\}$ and let $M = \max_{z \in C} |f(z)|$. Then for each integer $n \geq 0$,

$$\left| f^{(n)}(z_0) \right| \leq \frac{n!\, M}{r^n}$$

- **Liouville's Theorem:** The only bounded entire functions are the constant functions.

- **Maximum Modulus Principle:**
  - If f is analytic in a domain D and $|f(z)|$ achieves a maximum value at a point in D, then f is constant in D.
  - If f is analytic and non-constant in a domain D, then $|f(z)|$ has no maximum value in D.
  - A function analytic in a bounded domain and continuous up to and including its boundary attains its maximum modulus on the boundary.

- **Minimum Modulus Principle:** Let f be analytic in a bounded domain D and continuous up to and including its boundary. If f is nonzero in D, then the modulus $|f(z)|$ attains its minimum value on the boundary of D.

- If f has an isolated singularity at the point z_0 then the coefficient a_{-1} of $\dfrac{1}{z - z_0}$ in the Laurent expansion for f around z_0 is called the **residue** of f at z_0, denoted by $\text{Res}\,(f, z_0)$.

- **Methods to determine residues:**

(i) **Pole of order 1 (Simple pole).**

$$\text{Res }(f, z_0) = \lim_{z \to z_0} (z - z_0) f(z)$$

Furthermore, if $f(z) = \dfrac{P(z)}{Q(z)}$ with P and Q analytic at z_0, Q having a simple zero at z_0, and $P(z_0) \neq 0$, then

$$\text{Res }(f, z_0) = \frac{P(z_0)}{Q'(z_0)}$$

(ii) **Pole of order 2.**

$$\text{Res }(f, z_0) = \lim_{z \to z_0} \frac{d}{dz} \big((z - z_0)^2 f(z)\big)$$

(iii) **Pole of order 3.**

$$\text{Res }(f, z_0) = \frac{1}{2!} \cdot \lim_{z \to z_0} \frac{d^2}{dz^2} \big((z - z_0)^3 f(z)\big)$$

(iv) **Pole of order m.**

$$\text{Res }(f, z_0) = \frac{1}{(m-1)!} \cdot \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} \left((z - z_0)^m f(z)\right)$$

- **Residue Theorem:** If Γ is a simple closed positively oriented contour and $f(z)$ is analytic on and on Γ except at the points $z_1, z_2, \ldots, z_n$ inside Γ, then

$$\int_\Gamma f(z)\, dz = 2\pi i \sum_{k=1}^{n} \text{Res }(f, z_k)$$

- $f(z)$ is **meromorphic** in a domain D if at every point of D it is either analytic or has a pole.

- **Argument Principle:** If $f(z)$ is analytic and nonzero at each point of a simple closed positively oriented contour γ and is meromorphic inside γ, then

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\, dz = N_o - N_p$$

where

- N_o = the number of zeros of $f(z)$ inside γ, counted with multiplicity,
- N_p = the number of poles of $f(z)$ inside γ, counted with order.

Continued...

- **Rouché's Theorem:** If $f(z)$ and $h(z)$ are analytic inside and on a simple closed contour C and if $|h(z)| < |f(z)|$ holds at each point on C, then $f(z)$ and $f(z) + h(z)$ have the same total number of zeros, counting multiplicities, inside C.

- **Open Mapping Theorem:** If f is non-constant and analytic in a domain D, then its range $f(D)$ is an open set.

- Cauchy Principal Value of Improper Integral of Non-negative Function over $(-\infty, \infty)$

$$\int_{-\infty}^{\infty} f(x)\, dx = \lim_{n\to\infty} \int_{-n}^{n} f(x)\, dx$$

- **The Geometric series:** $\displaystyle\sum_{n=1}^{\infty} z^n = \begin{cases} \dfrac{1}{1-z} & , |z| < 1 \\ \infty & , |z| \geq 1 \end{cases}$

- **Comparison Test:** Suppose $|a_n| \leq b_n$ for all integers $n \geq N$. If the series $\sum_{n=0}^{\infty} b_n$ converges then $\sum_{n=0}^{\infty} a_n$ converges.

- **Ratio Test:** Suppose $\lim_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = L$.

  (i) If $L < 1$ then the series $\sum_{n=0}^{\infty} a_n$ converges.

  (ii) If $L > 1$ then the series $\sum_{n=0}^{\infty} a_n$ diverges.

- Let f be analytic at z_0. The series $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$ is called the Taylor series for f around z_0.

- If f is analytic in the disk $|z - z_0| < R$, then the Taylor series for f around z_0 converges to f for all z in this disk. Furthermore, the convergence of the series is uniform in any closed sub-disk $|z - z_0| \leq R' < R$. (The Taylor series will converge to f everywhere inside the largest open disk centered at z_0 over which f is analytic.)

- We may differentiate and integrate a Taylor series for f term wise. In particular, the Taylor series for $f \pm g$ is the term wise sum/difference of the Taylor series for f and g.

- For any power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ there is a number R (the *radius of convergence*) given by

$$R = \frac{1}{\limsup_{n\to\infty} |a_n|^{1/n}},$$

and, when the limit exists,

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|.$$

- The radius R depends only on the coefficients $\{a_n\}$, and has the following properties:
  (i) The series converges for $|z-z_0| < R$.
  (ii) The series converges uniformly in any closed sub-disk $|z-z_0| \leq R' < R$.
  (iii) The series diverges for $|z-z_0| > R$.

- If a power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges at a point having modulus r (i.e. at some z with $|z-z_0| = r$), then it converges at every point in the disk $|z-z_0| < r$.

- Let f_n be a sequence of functions analytic in a domain D and converging uniformly to f in compact subsets of D. Then f is analytic in D.

- A power series sums to a function that is analytic at every point inside its circle of convergence.

- Some important limits:
  (i) $\lim_{n\to\infty} \sqrt[n]{n!} = \infty$
  (ii) $\lim_{n\to\infty} n^{1/n} = 1$
  (iii) $\lim_{n\to\infty} \frac{\ln n}{n} = 0$
  (iv) $\lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n = e^x, \quad (x \in \mathbb{R})$

- Let f be analytic in the annulus $\leq r < |z-z_0| < R \leq \infty$. Then f can be expressed as the sum of two series, called the Laurent series,

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{\infty} a_{-n}(z-z_0)^{-n}$$

both series converge in the annulus, and converge uniformly in any closed sub-annulus $r < \rho_1 \leq |z - z_0| \leq \rho_2 < R$. The coefficients a_n are given by

$$a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z - z_0)^{n+1}} \, dz, \quad n \in \mathbb{Z}$$

where γ is any positively oriented simple closed contour lying in the annulus and containing z_0 in its interior.

- Laurent series expansion of an analytic function is unique.
- Let $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ and $\sum_{n=1}^{\infty} a_{-n}(z - z_0)^{-n}$ be two series with the following properties:

  (i) $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ converges for $|z - z_0| < R$,

  (ii) $\sum_{n=1}^{\infty} a_{-n}(z - z_0)^{-n}$ converges for $|z - z_0| > r$,

  (iii) $r < R$.

  Then there is a function $f(z)$, analytic for $r < |z - z_0| < R$, whose Laurent series in this annulus is given by $f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$.

- **Power series for sine and cosine:**

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n + 1)!} z^{2n+1}, \quad \text{and} \quad \cos z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n}.$$

- **Logarithm and exponential series:**

$$\log(1 + z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} z^n, \quad |z| < 1 \quad \text{and} \quad e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

- **Local Mapping Property:** Let $f(z)$ be analytic at z_0 and let $\omega_0 = f(z_0)$. Then the function $f(z) - \omega_0$ is analytic at z_0 and has a zero of order n at z_0. Consequently, for each $\varepsilon > 0$ there exists $\delta > 0$ such that for every $\omega \in \mathbb{C}$ with $|\omega - \omega_0| < \delta$ the function $f(z) - \omega$ has exactly n roots (counted with multiplicity) in the disk $|z - z_0| < \varepsilon$.
- **Weierstrass M-test:** Suppose $|f_n(z)| \leq M_n$ for all $z \in D$ and all n, and $\sum_{n=0}^{\infty} M_n$ converges. Then $\sum_{n=0}^{\infty} f_n(z)$ converges uniformly on D.

- f has a zero of order m at $z_0 \iff f$ is analytic at z_0 and

$$f^{(k)}(z_0) = \begin{cases} 0 & , k = 0, 1, \ldots, m-1, \\ \neq 0 & , k = m. \end{cases}$$

  Equivalently, $f(z) = (z - z_0)^m g(z)$ where g is analytic at z_0 and $g(z_0) \neq 0$.

- If f is an analytic function such that $f(z_0) = 0$ then either f is identically zero in a neighborhood of z_0 or there is a punctured disk about z_0 in which f has no zeros.

- An **isolated singularity** of f is a point z_0 such that f is analytic in some punctured disk $0 < |z - z_0| < R$ but not analytic at z_0 itself.

- Let f have an isolated singularity at a point z_0. The following are equivalent:

  (i) z_0 is a **removable singularity**

$$\begin{aligned} &\iff f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad \text{in some punctured disk } 0 < |z - z_0| < R \\ &\iff |f| \quad \text{is bounded in some punctured disk } 0 < |z - z_0| < R \\ &\iff \lim_{z \to z_0} f(z) \quad \text{exists and is finite} \\ &\iff f \quad \text{can be redefined at } z_0 \text{ so that } f \text{ is analytic at } z_0. \\ &\iff f(z_0) := \lim_{z \to z_0} f(z) \end{aligned}$$

  (ii) z_0 is a **pole of order** m

$$\iff \lim_{z \to z_0} |f(z)| = \infty$$

$$\iff f(z) = \sum_{n=-m}^{\infty} a_n (z - z_0)^n, \quad \text{with } a_{-m} \neq 0,$$

  in some punctured disk $0 < |z - z_0| < R$ of z_0

$$\iff f(z) = \frac{g(z)}{(z - z_0)^m} \quad \text{where } g(z) \text{ is analytic at } z_0 \text{ with } g(z_0) \neq 0$$

  (iii) z_0 is an **essential singularity**

$$\iff f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

  in some punctured disk $0 < |z - z_0| < R$, where $a_n \neq 0, \forall n$

$$\iff |f| \quad \text{is neither bounded near } z_0 \text{ nor tends to infinity as } z \to z_0$$

- **Picard's Theorem:** A function with an essential singularity assumes every complex number, with possibly one exception, as a value in any neighborhood of this singularity.

- If f has a zero of order m at z_0 then $\frac{1}{f}$ has a pole of order m at z_0.

- If f has a pole of order m at z_0 then $\frac{1}{f}$ has a removable singularity at z_0; if we define $\frac{1}{f}(z_0) = 0$ then $\frac{1}{f}$ has a zero of order m at z_0.

- $f(z)$ is **analytic at** $z = \infty$ (also called $f(z)$ has removable singularity at $z = \infty$)

  $\iff f\left(\frac{1}{z}\right)$ is analytic (or has a removable singularity) at $z = 0$.

  $\iff |f|$ is bounded for sufficiently large $|z|$, i.e. $\lim_{|z|\to\infty} |f(z)| = M < \infty$.

- $f(z)$ has a **pole of order** m **at** $z = \infty$

$$\iff f\left(\frac{1}{z}\right) \text{ has a pole of order } m \text{ at } z = 0$$
$$\iff \lim_{|z|\to\infty} |f(z)| = \infty.$$

- $f(z)$ has an **essential singularity at** $z = \infty$

  $\iff f\left(\frac{1}{z}\right)$ has an essential singularity at $z = 0$

  $\iff |f(z)|$ neither is bounded for large $|z|$nor goes to infinity as $z \to \infty$.

- **Identity Theorem:** Let $f(z)$ be analytic in a domain D. If the set $\{z \in D : f(z) = 0\}$ of zeros of $f(z)$ has a limit point in D, then $f \equiv 0$ in D.

- **Uniqueness Theorem:** Let $f(z)$ and $g(z)$ be analytic in a domain D. If the set $\{z \in D : f(z) = g(z)\}$ has a limit point in D, then $f \equiv g$ in D.

- **Schwarz's Lemma:** Let $f(z)$ be analytic on the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ satisfying $f(0) = 0$ and $|f(z)| \leq 1$ for all $z \in \mathbb{D}$. Then

  (i) $|f(z)| \leq |z|$ for all $z \in \mathbb{D}$,

  (ii) $|f'(0)| \leq 1$.

  Furthermore, if equality holds in (a) for some $z \neq 0$ or if equality holds in (b), then $f(z)$ is a rotation in $\mathbb{D}$, i.e. there is a constant $\alpha \in \mathbb{C}$ with $|\alpha| = 1$ such that $f(z) = \alpha z$ for all $z \in \mathbb{D}$.

Problems:

Q 1. Find the values of:

(a) $(1+2i)^3$

(b) $\dfrac{5}{-3+4i}$

(c) $\left(\dfrac{2+i}{3-2i}\right)^2$

Solution:

(a) Using the binomial expansion $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$:

$$\begin{aligned}(1+2i)^3 &= (1)^3 + 3(1)^2(2i) + 3(1)(2i)^2 + (2i)^3 \\ &= 1 + 6i + 12i^2 + 8i^3 \\ &= 1 + 6i - 12 - 8i \\ &= -11 - 2i\end{aligned}$$

(b) Multiply the numerator and denominator by the complex conjugate, $-3-4i$:

$$\begin{aligned}\frac{5}{-3+4i} &= \frac{5(-3-4i)}{(-3+4i)(-3-4i)} \\ &= \frac{-15-20i}{(-3)^2+(4)^2} \\ &= \frac{-15-20i}{9+16} \\ &= \frac{-15-20i}{25} \\ &= -\frac{3}{5} - \frac{4}{5}i\end{aligned}$$

(c) First, simplify the fraction inside the parentheses by multiplying by the conjugate, $3+2i$:

$$\begin{aligned}\frac{2+i}{3-2i} &= \frac{(2+i)(3+2i)}{(3-2i)(3+2i)} \\ &= \frac{6+4i+3i+2i^2}{3^2+2^2}\end{aligned}$$

$$= \frac{6 + 7i - 2}{9 + 4}$$
$$= \frac{4 + 7i}{13}$$

Now, square the result:

$$\left(\frac{4 + 7i}{13}\right)^2 = \frac{(4 + 7i)^2}{13^2}$$
$$= \frac{16 + 56i + 49i^2}{169}$$
$$= \frac{16 + 56i - 49}{169}$$
$$= -\frac{33}{169} + \frac{56}{169}i$$

Q 2. If $z = x + iy$, where x and y are real, find the real and imaginary parts of:

(a) $\frac{1}{z}$

(b) $\frac{z - 1}{z + 1}$

(c) $\frac{1}{z^2}$

Solution:

(a)

$$\frac{1}{x + iy} = \frac{1}{x + iy} \cdot \frac{x - iy}{x - iy}$$
$$= \frac{x - iy}{x^2 + y^2}$$
$$= \frac{x}{x^2 + y^2} + i\left(-\frac{y}{x^2 + y^2}\right)$$

Real part: $\frac{x}{x^2 + y^2}$, Imaginary part: $-\frac{y}{x^2 + y^2}$

(b)

$$\frac{(x - 1) + iy}{(x + 1) + iy} = \frac{(x - 1) + iy}{(x + 1) + iy} \cdot \frac{(x + 1) - iy}{(x + 1) - iy}$$

$$
\begin{aligned}
&= \frac{(x-1)(x+1) - iy(x-1) + iy(x+1) - i^2y^2}{(x+1)^2 + y^2} \\
&= \frac{x^2 - 1 + iyx + iy - iyx + iy + y^2}{(x+1)^2 + y^2} \\
&= \frac{x^2 + y^2 - 1}{(x+1)^2 + y^2} + i\left(\frac{2y}{(x+1)^2 + y^2}\right)
\end{aligned}
$$

Real part: $\dfrac{x^2 + y^2 - 1}{(x+1)^2 + y^2}$, Imaginary part: $\dfrac{2y}{(x+1)^2 + y^2}$

(c)

$$
\begin{aligned}
\frac{1}{(x+iy)^2} &= \frac{1}{x^2 - y^2 + i(2xy)} \\
&= \frac{1}{(x^2 - y^2) + i(2xy)} \cdot \frac{(x^2 - y^2) - i(2xy)}{(x^2 - y^2) - i(2xy)} \\
&= \frac{(x^2 - y^2) - i(2xy)}{(x^2 - y^2)^2 + (2xy)^2} \\
&= \frac{x^2 - y^2}{(x^2 + y^2)^2} + i\left(-\frac{2xy}{(x^2 + y^2)^2}\right)
\end{aligned}
$$

Real part: $\dfrac{x^2 - y^2}{(x^2 + y^2)^2}$, Imaginary part: $-\dfrac{2xy}{(x^2 + y^2)^2}$

Q 3. Show that $\left(\dfrac{-1 + i\sqrt{3}}{2}\right)^3 = 1$.

Solution:

$$
\begin{aligned}
\left(\frac{-1 + i\sqrt{3}}{2}\right)^3 &= \left(-\frac{1}{2}\right)^3 + 3\left(-\frac{1}{2}\right)^2\left(\frac{i\sqrt{3}}{2}\right) + 3\left(-\frac{1}{2}\right)\left(\frac{i\sqrt{3}}{2}\right)^2 + \left(\frac{i\sqrt{3}}{2}\right)^3 \\
&= -\frac{1}{8} + 3\left(\frac{1}{4}\right)\left(\frac{i\sqrt{3}}{2}\right) + 3\left(-\frac{1}{2}\right)\left(\frac{-3}{4}\right) + \frac{i^3(\sqrt{3})^3}{2^3} \\
&= -\frac{1}{8} + \frac{3i\sqrt{3}}{8} + \frac{9}{8} - \frac{3i\sqrt{3}}{8} \\
&= -\frac{1}{8} + \frac{9}{8} \\
&= 1
\end{aligned}
$$

Q 4. Verify by calculation that the values of $\dfrac{z}{z^2+1}$ for $z = x+iy$ and $z = x-iy$ are conjugate.

Solution: Let $f(z) = \dfrac{z}{z^2+1}$ and $z = x+iy$.
The denominator is:

$$z^2+1 = (x+iy)^2+1 = x^2-y^2+1+i(2xy)$$

Let $A = x^2-y^2+1$ and $B = 2xy$.
Then:

$$\begin{aligned} f(x+iy) &= \frac{x+iy}{A+iB} \\ &= \frac{(x+iy)(A-iB)}{A^2+B^2} \\ &= \frac{(xA+yB)+i(yA-xB)}{A^2+B^2} \end{aligned}$$

Now let $z = x-iy$. The denominator is:

$$(x-iy)^2+1 = x^2-y^2+1-i(2xy) = A-iB$$

Then:

$$\begin{aligned} f(x-iy) &= \frac{x-iy}{A-iB} \\ &= \frac{(x-iy)(A+iB)}{A^2+B^2} \\ &= \frac{(xA+yB)+i(xB-yA)}{A^2+B^2} \end{aligned}$$

Let $f(x+iy) = u+iv$, where, $u = \dfrac{xA+yB}{A^2+B^2}$ and $v = \dfrac{yA-xB}{A^2+B^2}$.
Therefore,

$$f(x-iy) = \frac{xA+yB}{A^2+B^2} - i\left(\frac{yA-xB}{A^2+B^2}\right) = u-iv$$

Since $f(x+iy) = u+iv$ and $f(x-iy) = u-iv$, the values are complex conjugates.

Q 5. Prove that $\left|\dfrac{a-b}{1-\overline{a}b}\right| = 1$ if either $|a| = 1$ or $|b| = 1$. What exception must be made if $|a| = |b| = 1$?

Solution: We know the property $|z|^2 = z\overline{z}$. Thus,

$$\left|\frac{a-b}{1-\overline{a}b}\right|^2 = \frac{a-b}{1-\overline{a}b} \cdot \overline{\left(\frac{a-b}{1-\overline{a}b}\right)} = \frac{a-b}{1-\overline{a}b} \cdot \frac{\overline{a}-\overline{b}}{1-a\overline{b}}$$

Expanding the numerator and denominator:

$$\frac{a\overline{a} - a\overline{b} - b\overline{a} + b\overline{b}}{1 - a\overline{b} - \overline{a}b + a\overline{a}b\overline{b}} = \frac{|a|^2 - a\overline{b} - \overline{a}b + |b|^2}{1 - a\overline{b} - \overline{a}b + |a|^2|b|^2}$$

Case 1: If $|a| = 1$, substituting $|a|^2 = 1$ into the expression:

$$\frac{1 - a\overline{b} - \overline{a}b + |b|^2}{1 - a\overline{b} - \overline{a}b + (1)|b|^2} = \frac{1 - a\overline{b} - \overline{a}b + |b|^2}{1 - a\overline{b} - \overline{a}b + |b|^2} = 1$$

Case 2: If $|b| = 1$, substituting $|b|^2 = 1$ into the expression:

$$\frac{|a|^2 - a\overline{b} - \overline{a}b + 1}{1 - a\overline{b} - \overline{a}b + |a|^2(1)} = \frac{|a|^2 - a\overline{b} - \overline{a}b + 1}{1 - a\overline{b} - \overline{a}b + |a|^2} = 1$$

In both cases, the square of the modulus is 1, so $\left|\dfrac{a-b}{1-\overline{a}b}\right| = 1$.

The Exception: The exception must be made when the denominator is zero. If $|a| = |b| = 1$, the expression $1 - \overline{a}b$ could be zero. Specifically, if $|a| = 1$ and $|b| = 1$, then $\overline{a} = 1/a$. The denominator becomes:

$$1 - \frac{b}{a}$$

This is zero if $a = b$.
The identity holds provided that $a \neq b$ when $|a| = |b| = 1$, otherwise the expression results in the indeterminate form $\dfrac{0}{0}$.

Q 6. Find the absolute values of:

(a) $-2i(3+i)(2+4i)(1+i)$

(b) $\dfrac{(3+4i)(-1+2i)}{(-1-i)(3-i)}$

Solution:

(a)

$$\begin{aligned}
|-2i(3+i)(2+4i)(1+i)| &= |-2i| \cdot |3+i| \cdot |2+4i| \cdot |1+i| \\
&= \sqrt{0^2 + (-2)^2} \cdot \sqrt{3^2 + 1^2} \cdot \sqrt{2^2 + 4^2} \cdot \sqrt{1^2 + 1^2} \\
&= 2 \cdot \sqrt{10} \cdot 2\sqrt{5} \cdot \sqrt{2} \\
&= 4 \cdot \sqrt{100} \\
&= 4 \cdot 10 \\
&= 40
\end{aligned}$$

(b)

$$\begin{aligned}
\left|\frac{(3+4i)(-1+2i)}{(-1-i)(3-i)}\right| &= \frac{|3+4i|\cdot|-1+2i|}{|-1-i|\cdot|3-i|} \\
&= \frac{\sqrt{3^2+4^2}\cdot\sqrt{(-1)^2+2^2}}{\sqrt{(-1)^2+(-1)^2}\cdot\sqrt{3^2+(-1)^2}} \\
&= \frac{5\cdot\sqrt{5}}{\sqrt{2}\cdot\sqrt{10}} \\
&= \frac{5\sqrt{5}}{\sqrt{20}} \\
&= \frac{5\sqrt{5}}{2\sqrt{5}} \\
&= \frac{5}{2}
\end{aligned}$$

Q 7. Given three vertices of a parallelogram z_1, z_2 and z_3, find the fourth vertex z_4, opposite to the vertex z_2.

Solution: In a parallelogram, the diagonals bisect each other. This implies that the midpoint of the diagonal connecting z_1 and z_3 must be the same as the midpoint of the diagonal connecting z_2 and z_4. The midpoint formula for complex numbers is:

$$M = \frac{z_a + z_b}{2}$$

Equating the midpoints of both diagonals;

$$\frac{z_1+z_3}{2} = \frac{z_2+z_4}{2} \implies z_1+z_3 = z_2+z_4 \implies z_4 = z_1+z_3-z_2$$

The fourth vertex z_4 of a parallelogram, which is opposite to the vertex z_2, is given by:

$$z_4 = z_1 + z_3 - z_2$$

Q 8. Prove that if $z_1 + z_2 + z_3 = 0$ and $|z_1| = |z_2| = |z_3| = 1$, then the points z_1, z_2 and z_3 are the vertices of an equilateral triangle inscribed in the unit circle.

Solution: Since $|z_1| = |z_2| = |z_3| = 1$, all three points lie on the unit circle. Consequently, the **circumcenter** of the triangle formed by these vertices is the origin $(0, 0)$. The centroid G of any triangle in the complex plane is given by:

$$G = \frac{z_1 + z_2 + z_3}{3}$$

Given $z_1 + z_2 + z_3 = 0$, the **centroid** is also at the origin. In any triangle, if the circum-center and the centroid coincide, the triangle must be equilateral.

Algebraic Proof: To prove the triangle is equilateral, we show that the distances between the vertices are equal. Let $L_{12} = |z_1 - z_2|$. Using the identity $|a-b|^2 = (a-b)(\overline{a}-\overline{b})$, we get;

$$|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - (z_1\overline{z}_2 + \overline{z}_1 z_2)$$

Since $|z_i| = 1$, we have $\overline{z}_i = 1/z_i$:

$$|z_1 - z_2|^2 = 2 - (z_1\overline{z}_2 + \overline{z}_1 z_2)$$

From the given condition $z_1 + z_2 + z_3 = 0$, we have $z_1 + z_2 = -z_3$. Then;

$$|z_1 + z_2|^2 = |-z_3|^2$$

$$\begin{aligned}
\implies (z_1 + z_2)(\overline{z}_1 + \overline{z}_2) &= 1 \\
\implies |z_1|^2 + |z_2|^2 + z_1\overline{z}_2 + \overline{z}_1 z_2 &= 1 \\
\implies 1 + 1 + (z_1\overline{z}_2 + \overline{z}_1 z_2) &= 1 \\
\implies z_1\overline{z}_2 + \overline{z}_1 z_2 &= -1
\end{aligned}$$

Substituting this back into the distance formula:

$$|z_1 - z_2|^2 = 2 - (-1) = 3 \implies |z_1 - z_2| = \sqrt{3}$$

By symmetry, $|z_2 - z_3| = \sqrt{3}$ and $|z_3 - z_1| = \sqrt{3}$. Since all side lengths are equal, the triangle is **equilateral**.

Q 9. Explain the geometric meaning of the relations:

(a) $\text{Im}\left(\dfrac{z - z_1}{z - z_2}\right) = 0$

(b) $\text{Re}\left(\dfrac{z - z_1}{z - z_2}\right) = 0$

Solution: The relations involve the positions of points z, z_1, and z_2 in the complex plane. Let z, z_1, and z_2 be represented by points P, A, and B respectively.

(a) $\text{Im}\left(\dfrac{z - z_1}{z - z_2}\right) = 0$
Meaning: The points z, z_1, and z_2 are **collinear**.

Explanation: For the imaginary part of a complex number to be zero, the number must be purely real. This implies that the argument (angle) of the ratio is either 0 or π:

$$\arg\left(\frac{z - z_1}{z - z_2}\right) = n\pi, \quad n \in \mathbb{Z}$$

Geometrically, this means the vector $\overrightarrow{BP}$ is parallel to the vector $\overrightarrow{AP}$. Thus, the point z lies on the straight line passing through z_1 and z_2 (where $z \neq z_2$).

(b) $\text{Re}\left(\frac{z - z_1}{z - z_2}\right) = 0$
Meaning: The point z lies on a **circle** with the segment z_1z_2 as its diameter.
Explanation: For the real part of a complex number to be zero, the number must be purely imaginary. This implies that the argument is an odd multiple of $\pi/2$:

$$\arg\left(\frac{z - z_1}{z - z_2}\right) = \pm\frac{\pi}{2}$$

Geometrically, this means the vectors $(z - z_1)$ and $(z - z_2)$ are perpendicular $(\overrightarrow{AP} \perp \overrightarrow{BP})$. By Thales' Theorem, the locus of a point P that forms a right angle with two fixed points A and B is a circle with diameter AB (where $z \neq z_2$).

Q 10. Simplify:

(a) $1 + \cos\phi + \cos 2\phi + \cdots + \cos n\phi$

(b) $\sin\phi + \sin 2\phi + \cdots + \sin n\phi$

Solution: To simplify these sums, we treat them as the real and imaginary parts of a geometric series using Euler's formula: $e^{ik\phi} = \cos(k\phi) + i\sin(k\phi)$. Let the sum be:

$$S = \sum_{k=0}^{n} e^{ik\phi} = 1 + e^{i\phi} + e^{i2\phi} + \cdots + e^{in\phi}$$

This is a geometric progression with first term $a = 1$, common ratio $r = e^{i\phi}$, and $n + 1$ terms. Then,

$$\begin{aligned} S &= \frac{1 - e^{i(n+1)\phi}}{1 - e^{i\phi}} \\ &= \frac{e^{i(n+1)\phi/2}\left(e^{-i(n+1)\phi/2} - e^{i(n+1)\phi/2}\right)}{e^{i\phi/2}\left(e^{-i\phi/2} - e^{i\phi/2}\right)} \end{aligned}$$

$$= e^{in\phi/2} \frac{-2i \sin\left(\dfrac{(n+1)\phi}{2}\right)}{-2i \sin\left(\dfrac{\phi}{2}\right)}$$

$$= \left(\cos\left(\frac{n\phi}{2}\right) + i \sin\left(\frac{n\phi}{2}\right)\right) \frac{\sin\left(\dfrac{(n+1)\phi}{2}\right)}{\sin\left(\dfrac{\phi}{2}\right)}$$

Distributing the fraction across the real and imaginary components:

(a) **Cosine Sum (Real Part):**

$$1 + \cos\phi + \cdots + \cos n\phi = \frac{\cos\left(\dfrac{n\phi}{2}\right) \sin\left(\dfrac{(n+1)\phi}{2}\right)}{\sin\left(\dfrac{\phi}{2}\right)}$$

(b) **Sine Sum (Imaginary Part):**

$$\sin\phi + \sin 2\phi + \cdots + \sin n\phi = \frac{\sin\left(\dfrac{n\phi}{2}\right) \sin\left(\dfrac{(n+1)\phi}{2}\right)}{\sin\left(\dfrac{\phi}{2}\right)}$$

Q 11. Verify Cauchy-Riemann's equations for the functions z^2 and z^3.

Solution: To verify the Cauchy-Riemann equations for $f(z) = u + iv$, we check:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

For the function $f(z) = z^2$. Substitute $z = x + iy$:

$$f(z) = (x+iy)^2 = x^2 + 2ixy - y^2 = (x^2 - y^2) + i(2xy)$$

Thus, $u = x^2 - y^2$ and $v = 2xy$. Then the partial derivatives are:

$$u_x = \frac{\partial}{\partial x}(x^2 - y^2) = 2x \quad , \quad v_x = \frac{\partial}{\partial x}(2xy) = 2y$$
$$u_y = \frac{\partial}{\partial y}(x^2 - y^2) = -2y \quad , \quad v_y = \frac{\partial}{\partial y}(2xy) = 2x$$

Continued...

Therefore $u_x = v_y$ and $u_y = -v_x$. Hence, the Cauchy-Riemann equations are satisfied by the function z^2.

For the function $f(z) = z^3$. Substitute $z = x + iy$:

$$f(z) = (x+iy)^3 = x^3 + 3x^2(iy) + 3x(iy)^2 + (iy)^3 = (x^3 - 3xy^2) + i(3x^2y - y^3)$$

Thus, $u = x^3 - 3xy^2$ and $v = 3x^2y - y^3$. Then the partial derivatives are:

$$\begin{aligned} u_x &= \frac{\partial}{\partial x}(x^3 - 3xy^2) = 3x^2 - 3y^2 \quad , \quad v_x = \frac{\partial}{\partial x}(3x^2y - y^3) = 6xy \\ u_y &= \frac{\partial}{\partial y}(x^3 - 3xy^2) = -6xy \quad , \quad v_y = \frac{\partial}{\partial y}(3x^2y - y^3) = 3x^2 - 3y^2 \end{aligned}$$

Therefore $u_x = v_y$ and $u_y = -v_x$. Hence, the Cauchy-Riemann equations are satisfied by the function z^3.

Q 12. Show that an analytic function cannot have a constant absolute value without reducing to a constant.

Solution: Let $f(z) = u(x,y) + iv(x,y)$ be an analytic function on a connected domain D. Suppose $|f(z)| = c$ for all $z \in D$, where c is a real constant. If $c = 0$, then $u^2 + v^2 = 0$. Since $u, v \in \mathbb{R}$, this implies $u = 0$ and $v = 0$, so $f(z) = 0$ is a constant. Now assume $c \neq 0$. Then $u^2 + v^2 = c^2$. Differentiating both sides with respect to x and y gives:

$$2uu_x + 2vv_x = 0 \implies uu_x + vv_x = 0 \tag{1}$$
$$2uu_y + 2vv_y = 0 \implies uu_y + vv_y = 0 \tag{2}$$

Using the Cauchy-Riemann equations ($u_x = v_y$ and $u_y = -v_x$), we substitute into (1) and (2):

$$\begin{aligned} uu_x - vu_y &= 0 \\ vu_x + uu_y &= 0 \end{aligned}$$

This represents a linear system for u_x and u_y. We can write it in matrix form as:

$$\begin{pmatrix} u & -v \\ v & u \end{pmatrix} \begin{pmatrix} u_x \\ u_y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The determinant of the coefficient matrix is $u^2 + v^2 = c^2$. Since we assumed $c \neq 0$, the determinant is non-zero. The only solution to this system is the trivial solution:

$$u_x = 0 \quad \text{and} \quad u_y = 0$$

By the Cauchy-Riemann equations, it follows that $v_y = u_x = 0$ and $v_x = -u_y = 0$. Since all partial derivatives of u and v are zero on a connected domain, u and v must be constants. Therefore, $f(z) = u + iv$ is constant.

Q 13. Show that a harmonic function satisfies the differential equation $\dfrac{\partial^2 u}{\partial z \partial \overline{z}} = 0$.

Solution: To show that a harmonic function $u(x, y)$ satisfies the differential equation $\dfrac{\partial^2 u}{\partial z \partial \overline{z}} = 0$, we utilize the definitions of the Wirtinger derivatives.

Let $z = x + iy$ and $\overline{z} = x - iy$. The partial derivative operators with respect to z and $\overline{z}$ are defined as:

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

Appling these operators sequentially to the function u, we get;

$$\begin{aligned}
\frac{\partial^2 u}{\partial z \partial \overline{z}} &= \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial \overline{z}}\right) \\
&= \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\left[\frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right)\right] \\
&= \frac{1}{4}\left(\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right) - i\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right)\right) \\
&= \frac{1}{4}\left(\frac{\partial^2 u}{\partial x^2} + i\frac{\partial^2 u}{\partial x \partial y} - i\frac{\partial^2 u}{\partial y \partial x} - i^2\frac{\partial^2 u}{\partial y^2}\right)
\end{aligned}$$

Assuming u is C^2 (smooth enough that mixed partials commute) and using $i^2 = -1$, we have;

$$\frac{\partial^2 u}{\partial z \partial \overline{z}} = \frac{1}{4}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

We know that a function u is harmonic if it satisfies Laplace's equation:

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Substituting this into our derived expression:

$$\frac{\partial^2 u}{\partial z \partial \overline{z}} = \frac{1}{4}(0) = 0$$

Thus, for any harmonic function u, the equation $\dfrac{\partial^2 u}{\partial z \partial \overline{z}} = 0$ is satisfied.

Q 14. Find the value of:

(a) $\sin i$

(b) $\cos i$

Solution:

(a) We use the complex definition of the sine function:

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

Then;

$$\sin i = \frac{e^{i(i)} - e^{-i(i)}}{2i} = \frac{e^{-1} - e^{1}}{2i} = \frac{i(e^{-1} - e)}{-2} = i\left(\frac{e^{1} - e^{-1}}{2}\right) = i \sinh 1$$

(b) We use the complex definition of the cosine function:

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

Then;

$$\cos i = \frac{e^{i(i)} + e^{-i(i)}}{2} = \frac{e^{-1} + e^{1}}{2} = \cosh 1$$

Q 15. Find the value of e^z for:

(a) $z = \dfrac{-\pi i}{2}$

(b) $z = \dfrac{3\pi i}{4}$

(c) $z = \dfrac{2\pi i}{3}$

Solution: To evaluate e^z for the given values of z, we use **Euler's Formula**:

$$e^{ix} = \cos(x) + i\sin(x)$$

(a) **For** $z = \dfrac{-\pi i}{2}$**:**

$$e^{-\pi i/2} = \cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right) = 0 + i(-1) = -i$$

(b) **For** $z = \frac{3\pi i}{4}$**:**

$$e^{3\pi i/4} = \cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$$

(c) **For** $z = \frac{2\pi i}{3}$**:**

$$e^{2\pi i/3} = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right) = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$

Q 16. Find the value of z when:

(a) $e^z = 2$

(b) $e^z = -1$

(c) $e^z = i$

(d) $e^z = -\frac{i}{2}$

(e) $e^z = -1 - i$

(f) $e^z = 1 + 2i$

Solution: The general solution for the complex equation $e^z = w$ is given by:

$$z = \ln|w| + i(\text{Arg}(w) + 2k\pi)$$

where k is any integer ($k \in \mathbb{Z}$) and $\text{Arg}(w)$ is the principal argument.

(a) $e^z = 2$
Magnitude: $|2| = 2$
Argument: $\text{Arg}(2) = 0$
Therefore, $z = \ln(2) + 2k\pi i$

(b) $e^z = -1$
Magnitude: $|-1| = 1$
Argument: $\text{Arg}(-1) = \pi$
Therefore, $z = \ln(1) + i(\pi + 2k\pi) = (2k+1)\pi i$

(c) $e^z = i$
Magnitude: $|i| = 1$
Argument: $\text{Arg}(i) = \frac{\pi}{2}$
Therefore, $z = \ln(1) + i\left(\frac{\pi}{2} + 2k\pi\right) = \left(2k + \frac{1}{2}\right)\pi i$

(d) $e^z = -\frac{i}{2}$
Magnitude: $|-\frac{i}{2}| = \frac{1}{2}$
Argument: $\text{Arg}(-\frac{i}{2}) = -\frac{\pi}{2}$
Therefore, $z = \ln\left(\frac{1}{2}\right) + i\left(-\frac{\pi}{2} + 2k\pi\right) = -\ln(2) + \left(2k - \frac{1}{2}\right)\pi i$

(e) $e^z = -1 - i$
Magnitude: $|-1-i| = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$
Argument: $\text{Arg}(-1-i) = -\frac{3\pi}{4}$
Therefore, $z = \ln(\sqrt{2}) + i\left(-\frac{3\pi}{4} + 2k\pi\right) = \frac{1}{2}\ln(2) + \left(2k - \frac{3}{4}\right)\pi i$

(f) $e^z = 1 + 2i$
Magnitude: $|1+2i| = \sqrt{1^2 + 2^2} = \sqrt{5}$
Argument: $\text{Arg}(1+2i) = \arctan(2)$
Therefore, $z = \ln(\sqrt{5}) + i(\arctan(2) + 2k\pi) = \frac{1}{2}\ln(5) + i(\arctan(2) + 2k\pi)$

Q 17. Find the real and imaginary parts of z^z.

Solution: Let $z = x + iy$. In polar form, $z = |z|e^{i\theta}$, where $|z| = \sqrt{x^2+y^2}$ and $\theta = \arg(z)$. The expression z^z is defined as:

$$z^z = e^{z\ln z}$$

Using the principal value of the complex logarithm, $\ln z = \ln|z| + i\theta$, we expand the exponent:

$$\begin{aligned} z\ln z &= (x+iy)(\ln|z| + i\theta) \\ &= x\ln|z| + ix\theta + iy\ln|z| + i^2 y\theta \\ &= (x\ln|z| - y\theta) + i(x\theta + y\ln|z|) \end{aligned}$$

Substituting this back into the exponential form:

$$z^z = e^{x\ln|z| - y\theta} \cdot e^{i(x\theta + y\ln|z|)}$$

Applying Euler's formula, $e^{ia} = \cos(a) + i\sin(a)$, we separate the real and imaginary components, Where $|z| = \sqrt{x^2 + y^2}$ and $\theta = \arctan\left(\frac{y}{x}\right)$. Thus,

$$\begin{aligned} \operatorname{Re}(z^z) &= e^{x\ln|z| - y\theta}\cos(x\theta + y\ln|z|) \\ \operatorname{Im}(z^z) &= e^{x\ln|z| - y\theta}\sin(x\theta + y\ln|z|) \end{aligned}$$

Q 18. Show that the distance $d(z, z')$ between the stereographic projections of $z \in \mathbb{C}$ and $z' \in \mathbb{C}$ satisfies $d(z, z') = \dfrac{2|z - z'|}{\sqrt{(1 + |z|^2)(1 + |z'|^2)}}$.

Solution: Consider the unit sphere $S^2 \subset \mathbb{R}^3$ defined by $x_1^2 + x_2^2 + x_3^2 = 1$. The inverse stereographic projection $\phi : \mathbb{C} \to S^2 \setminus \{(0, 0, 1)\}$ maps a point $z = x + iy$ to a point $P(x_1, x_2, x_3)$ where:

$$x_1 = \frac{2\operatorname{Re}(z)}{|z|^2 + 1}, \quad x_2 = \frac{2\operatorname{Im}(z)}{|z|^2 + 1}, \quad x_3 = \frac{|z|^2 - 1}{|z|^2 + 1}$$

Let $P = (x_1, x_2, x_3)$ and $P' = (x_1', x_2', x_3')$ be the images of z and z'. The squared Euclidean distance is:

$$\begin{aligned} d(z, z')^2 &= (x_1 - x_1')^2 + (x_2 - x_2')^2 + (x_3 - x_3')^2 \\ &= (x_1^2 + x_2^2 + x_3^2) + ({x_1'}^2 + {x_2'}^2 + {x_3'}^2) - 2(x_1x_1' + x_2x_2' + x_3x_3') \\ &= 1 + 1 - 2(x_1x_1' + x_2x_2' + x_3x_3') \\ &= 2 - 2(x_1x_1' + x_2x_2' + x_3x_3') \end{aligned}$$

Substituting the coordinates in terms of z and z':

$$x_1x_1' + x_2x_2' = \frac{4(\operatorname{Re}(z)\operatorname{Re}(z') + \operatorname{Im}(z)\operatorname{Im}(z'))}{(1 + |z|^2)(1 + |z'|^2)} = \frac{2(z\bar{z}' + \bar{z}z')}{(1 + |z|^2)(1 + |z'|^2)}$$

$$x_3x_3' = \frac{(|z|^2 - 1)(|z'|^2 - 1)}{(1 + |z|^2)(1 + |z'|^2)}$$

Substituting these into the distance formula:

$$\begin{aligned} d(z, z')^2 &= 2 - 2\left[\frac{2(z\bar{z}' + \bar{z}z') + |z|^2|z'|^2 - |z|^2 - |z'|^2 + 1}{(1 + |z|^2)(1 + |z'|^2)}\right] \\ &= \frac{2(1 + |z|^2 + |z'|^2 + |z|^2|z'|^2) - 2(2z\bar{z}' + 2\bar{z}z' + |z|^2|z'|^2 - |z|^2 - |z'|^2 + 1)}{(1 + |z|^2)(1 + |z'|^2)} \\ &= \frac{4|z|^2 + 4|z'|^2 - 4(z\bar{z}' + \bar{z}z')}{(1 + |z|^2)(1 + |z'|^2)} \\ &= \frac{4|z - z'|^2}{(1 + |z|^2)(1 + |z'|^2)} \end{aligned}$$

Therefore,

$$d(z, z') = \frac{2|z - z'|}{\sqrt{(1 + |z|^2)(1 + |z'|^2)}}$$

Q 19. Where does the function $f(z) = z\,\mathrm{Re}(z) + \overline{z}\,\mathrm{Im}(z) + \overline{z}$ have a complex derivative? Compute the derivative where it exists.

Solution: Let $z = x + iy$. Then $\mathrm{Re}(z) = x$, $\mathrm{Im}(z) = y$, and $\overline{z} = x - iy$. We rewrite $f(z)$ in terms of x and y:

$$\begin{aligned} f(z) &= (x + iy)x + (x - iy)y + (x - iy) \\ &= x^2 + ixy + xy - iy^2 + x - iy \\ &= (x^2 + xy + x) + i(xy - y^2 - y) \end{aligned}$$

Let $u(x, y) = x^2 + xy + x$ and $v(x, y) = xy - y^2 - y$. For $f(z)$ to have a complex derivative, the Cauchy-Riemann equations must be satisfied:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Computing the partial derivatives:

$$\begin{aligned} u_x &= 2x + y + 1 & v_x &= y \\ u_y &= x & v_y &= x - 2y - 1 \end{aligned}$$

Setting up the system of equations:

$$2x + y + 1 = x - 2y - 1 \implies x + 3y = -2 \tag{1}$$

$$x = -y \tag{2}$$

Substituting equation (2) into the equation (1):

$$-y + 3y = -2 \implies 2y = -2 \implies y = -1$$

This gives $x = 1$. Thus, the derivative exists only at $z = 1 - i$.
The derivative at this point is:

$$\begin{aligned} f'(1 - i) &= u_x(1, -1) + iv_x(1, -1) \\ &= [2(1) + (-1) + 1] + i(-1) \\ &= 2 - i \end{aligned}$$

Therefore, the function is differentiable only at $z = 1 - i$, and $f'(1 - i) = 2 - i$.

Q 20. Prove that if $z_1 + z_2 + z_3 + z_4 = 0$ and $|z_1| = |z_2| = |z_3| = |z_4| = 1$, then the points z_1, z_2, z_3 and z_4 are the vertices of an rectangle inscribed in the unit circle.

Solution: Rearranging the equilibrium equation:

$$z_1 + z_2 = -(z_3 + z_4) \tag{1}$$

Taking the squared magnitude of both sides:

$$|z_1 + z_2|^2 = |-(z_3 + z_4)|^2$$

Using $|z_i|^2 = z_i\overline{z}_i = 1$, we expand:

$$\begin{aligned}
\implies (z_1 + z_2)(\overline{z}_1 + \overline{z}_2) &= (z_3 + z_4)(\overline{z}_3 + \overline{z}_4) \\
\implies z_1\overline{z}_1 + z_2\overline{z}_2 + z_1\overline{z}_2 + z_2\overline{z}_1 &= z_3\overline{z}_3 + z_4\overline{z}_4 + z_3\overline{z}_4 + z_4\overline{z}_3 \\
\implies 1 + 1 + z_1\overline{z}_2 + z_2\overline{z}_1 &= 1 + 1 + z_3\overline{z}_4 + z_4\overline{z}_3 \\
\implies z_1\overline{z}_2 + \overline{z_1\overline{z}_2} &= z_3\overline{z}_4 + \overline{z_3\overline{z}_4}
\end{aligned}$$

Since $w + \overline{w} = 2\,\text{Re}(w)$, and for $|z_i| = 1$ the real part is the cosine of the angle between them, we have:

$$\text{Re}(z_1\bar{z}_2) = \text{Re}(z_3\bar{z}_4) \implies cos(\theta_{12}) = \cos(\theta_{34})$$

This implies $\theta_{12} = \pm\theta_{34}$.
By symmetry, we can similarly show $\text{Re}(z_1\overline{z}_3) = \text{Re}(z_2\overline{z}_4)$. These relations force the points to be paired such that $z_i = -z_j$. For instance, if $z_1 = -z_3$ and $z_2 = -z_4$:

- The segments $[z_1, z_3]$ and $[z_2, z_4]$ are diameters of the circle (passing through the origin).
- A quadrilateral whose diagonals are diameters of its circumscribed circle is a **rectangle**.

Q 21. Show that the union of two domains is a domain iff they have a common point.

Solution: A *domain* $D \subseteq \mathbb{C}$ (or $\mathbb{R}^n$) is a non-empty, open, and connected set.
Let D_1 and D_2 be domains. Then, we show that $D_1 \cup D_2$ is a domain if and only if $D_1 \cap D_2 \neq \emptyset$.
($\Rightarrow$) Assume $D_1 \cup D_2$ is a domain. By definition, a domain must be connected. Assume that $D_1 \cap D_2 = \emptyset$. Since D_1 and D_2 are non-empty open sets, their union $D_1 \cup D_2$ would be a disconnected set as, it is the union of two disjoint, non-empty open sets. This contradicts the assumption that the union is a domain. Thus, $D_1 \cap D_2 \neq \emptyset$.

($\Leftarrow$) Assume $D_1 \cap D_2 \neq \emptyset$.
Since D_1 and D_2 are domains, they are open. The union of any collection of open sets is open; therefore, $D_1 \cup D_2$ is open.
Let $p \in D_1 \cap D_2$. For any two points $x, y \in D_1 \cup D_2$:

- If both x, y are in D_1 (or both in D_2), they can be connected by a path because D_1 (or D_2) is connected.
- If $x \in D_1$ and $y \in D_2$, there exists a path $\gamma_1 \subset D_1$ from x to p and a path $\gamma_2 \subset D_2$ from p to y. The concatenated path $\gamma_1 \cup \gamma_2$ is contained in $D_1 \cup D_2$ and connects x to y.

Since every pair of points can be joined by a path, $D_1 \cup D_2$ is path-connected, which implies it is connected. Since $D_1 \cup D_2$ is both open and connected, it is a domain.

Q 22. Prove that every bounded sequence of complex numbers has a convergent subsequence.

Solution: Let $\{z_n\}_{n=1}^{\infty}$ be a bounded sequence in $\mathbb{C}$. We can write each term as $z_n = x_n + iy_n$, where $x_n, y_n \in \mathbb{R}$.
Since $\{z_n\}$ is bounded, there exists $M > 0$ such that $|z_n| \leq M$ for all $n \in \mathbb{N}$. Because $|x_n| \leq |z_n|$ and $|y_n| \leq |z_n|$, it follows that the real sequences $\{x_n\}$ and $\{y_n\}$ are also bounded in $\mathbb{R}$.
By the Bolzano-Weierstrass theorem for real numbers, the bounded sequence $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$ that converges to some $x \in \mathbb{R}$.
Now consider the corresponding subsequence of imaginary parts, $\{y_{n_k}\}$. Since $\{y_n\}$ is bounded, $\{y_{n_k}\}$ is also bounded. Applying the Bolzano-Weierstrass theorem again, there exists a further subsequence $\{y_{n_{k_j}}\}$ that converges to some $y \in \mathbb{R}$.
Now consider the subsequence $\{z_{n_{k_j}}\}$ of our original sequence. Since $\{x_{n_{k_j}}\}$ is a subsequence of the convergent sequence $\{x_{n_k}\}$, it also converges to x. Thus:

$$\lim_{j\to\infty} z_{n_{k_j}} = \lim_{j\to\infty} (x_{n_{k_j}} + iy_{n_{k_j}}) = x + iy$$

Since $x + iy \in \mathbb{C}$, the subsequence $\{z_{n_{k_j}}\}$ converges.

Q 23. If $E_1 \supset E_2 \supset E_3 \supset \dots$ is a decreasing sequence of nonempty compact sets, then $\bigcap_{n=1}^{\infty} E_n$ is not empty (Cantor's Lemma). Show by example that this need not be true if the sets are merely closed.

Solution: Consider the sequence of sets $E_n = [n, \infty)$ for $n = 1, 2, 3, \dots$.

- **Nonempty:** Each E_n is nonempty since $n \in E_n$.
- **Closed:** Each E_n is a closed ray in $\mathbb{R}$.
- **Nested:** Since $n < n+1$, we have $[n+1, \infty) \subset [n, \infty)$, so $E_1 \supset E_2 \supset E_3 \supset \dots$.
- **Empty Intersection:** Assume there exists some $x \in \bigcap_{n=1}^{\infty} E_n$. This would mean $x \geq n$ for every $n \in \mathbb{N}$. However, by the Archimedean property of the real

numbers, no such x exists. Thus:

$$\bigcap_{n=1}^{\infty}[n,\infty)=\emptyset$$

Q 24. Suppose that $f(z)$ is analytic and satisfies the condition $|f(z)^2-1|<1$ in a domain Ω. Show that either $\mathrm{Re}(f(z))>0$ or $\mathrm{Re}(f(z))<0$ throughout Ω.

Solution: Let $f(z)$ be an analytic function in a domain Ω satisfying the condition $|f(z)^2-1|<1$.
Let $w=f(z)^2$. The condition $|w-1|<1$ states that w lies within an open disk of radius 1 centered at $(1,0)$ in the complex plane. For any complex number $w=x+iy$ in this disk, we must have $x>0$. Therefore:

$$\mathrm{Re}(f(z)^2)>0 \quad \text{for all } z\in\Omega \tag{1}$$

Let $f(z)=u(z)+iv(z)$, where $u(z)=\mathrm{Re}(f(z))$ and $v(z)=\mathrm{Im}(f(z))$. Then:

$$f(z)^2=(u+iv)^2=(u^2-v^2)+i(2uv)$$

The real part is given by:

$$\mathrm{Re}(f(z)^2)=u^2-v^2$$

Then by equation (1), we have:

$$u^2-v^2>0 \implies u^2>v^2\geq 0$$

The inequality $u^2>v^2$ implies that $u^2>0$. This means that $u(z)\neq 0$ for any $z\in\Omega$. Specifically:

$$\mathrm{Re}(f(z))\neq 0 \quad \forall z\in\Omega$$

Since $f(z)$ is analytic, its real part $u(z)$ is a continuous function on the domain Ω. A domain in the complex plane is, by definition, a connected open set.
By the Intermediate Value Theorem, if a continuous function on a connected set is never zero, it must maintain a constant sign. Therefore, either:

$$\mathrm{Re}(f(z))>0, \quad \text{or} \quad \mathrm{Re}(f(z))<0, \quad \text{throughout } \Omega$$

Q 25. Show that every compact set is complete.

Solution: Let K be a compact subset of $\mathbb{C}$. To show that K is complete, we must prove that every Cauchy sequence in K converges to a limit in K.

- Let $\{z_n\}_{n=1}^{\infty}$ be a Cauchy sequence in K.

- Since K is compact, it is sequentially compact. This means every sequence in K has a subsequence $\{z_{n_k}\}$ that converges to some limit $L \in K$.
- We now show that the entire sequence $\{z_n\}$ converges to L. Since $\{z_n\}$ is Cauchy, for any $\epsilon > 0$, there exists an integer N such that:

$$|z_n - z_m| < \frac{\epsilon}{2} \quad \text{for all } n, m > N$$

- Since $z_{n_k} \to L$, we can choose a k large enough such that $n_k > N$ and:

$$|z_{n_k} - L| < \frac{\epsilon}{2}$$

- By the triangle inequality, for any $n > N$:

$$|z_n - L| \leq |z_n - z_{n_k}| + |z_{n_k} - L| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

- Thus, $z_n \to L$. Since $L \in K$, every Cauchy sequence in K converges to a point in K.

Therefore, K is complete.

Q 26. Prove that a set $S \subset \mathbb{C}$ is compact iff it is closed and bounded.

Solution: ($\implies$) Assume S is compact.

- **Boundedness:** Consider the open cover $\{B(0, n) : n \in \mathbb{N}\}$. Since S is compact, there exists a finite subcover $B(0, n_1), \ldots, B(0, n_k)$. Let $N = \max(n_1, \ldots, n_k)$. Then $S \subset B(0, N)$, so S is bounded.
- **Closedness:** Let $z \in S^c$. For each $w \in S$, choose disjoint open neighborhoods U_w of w and V_w of z. The collection $\{U_w\}_{w \in S}$ covers S. By compactness, a finite subcover $U_{w_1}, \ldots, U_{w_n}$ exists. The intersection $V = \bigcap_{i=1}^{n} V_{w_i}$ is an open neighborhood of z disjoint from S. Thus S^c is open, so S is closed.

($\impliedby$) Assume S is closed and bounded.

- Since S is bounded, $S \subset [-M, M] \times [-M, M] = R$ for some $M > 0$.
- In $\mathbb{R}^n$, any closed cell (rectangle) R is compact.
- S is a closed subset of the compact set R. Since any closed subset of a compact space is itself compact, S is compact.

Q 27. Let $z_1, z_2, z_3, z_4 \in \mathbb{C}$ be distinct points. Find point(s) of intersection of two lines l_1 and l_2 if $z_1 \in l_1$, $z_2 \in l_1$, $z_3 \in l_2$ and $z_4 \in l_2$,

Solution: Let $z_1, z_2, z_3, z_4 \in \mathbb{C}$ be distinct points. We want to find the intersection point z of the lines l_1 (passing through z_1 and z_2) and l_2 (passing through z_3 and z_4). A line passing through two points a and b is defined by the condition that for any point z on the line, the quotient $\frac{z-a}{b-a}$ is real. This implies:

$$\frac{z-a}{b-a} = \overline{\left(\frac{z-a}{b-a}\right)} = \frac{\overline{z}-\overline{a}}{\overline{b}-\overline{a}}$$

Rearranging this gives the standard complex form:

$$z(\overline{a}-\overline{b}) - \overline{z}(a-b) + a\overline{b} - b\overline{a} = 0$$

For our two lines, we have the following system:

$$z(\overline{z}_1 - \overline{z}_2) - \overline{z}(z_1 - z_2) = z_2\overline{z}_1 - z_1\overline{z}_2 \quad (1)$$

$$z(\overline{z}_3 - \overline{z}_4) - \overline{z}(z_3 - z_4) = z_4\overline{z}_3 - z_3\overline{z}_4 \quad (2)$$

Solving the system using Cramer's Rule the intersection point z is:

$$z = \frac{(z_1\overline{z}_2 - z_2\overline{z}_1)(z_3 - z_4) - (z_3\overline{z}_4 - z_4\overline{z}_3)(z_1 - z_2)}{(z_1 - z_2)(\overline{z}_3 - \overline{z}_4) - (z_3 - z_4)(\overline{z}_1 - \overline{z}_2)}$$

Q 28. If $T_1 z = \frac{z-2}{z+3}$ and $T_2 z = \frac{z}{z+1}$, find T_1T_2z, T_2T_1z and $T_1^{-1}T_2z$.

Solution: Given:

$$T_1(z) = \frac{z-2}{z+3}, \quad T_2(z) = \frac{z}{z+1}$$

- T_1T_2z:

$$T_1(T_2(z)) = \frac{\left(\frac{z}{z+1}\right) - 2}{\left(\frac{z}{z+1}\right) + 3} = \frac{z - 2(z+1)}{z + 3(z+1)} = \frac{z - 2z - 2}{z + 3z + 3} = \frac{-z-2}{4z+3}$$

- T_2T_1z:

$$T_2(T_1(z)) = \frac{\left(\frac{z-2}{z+3}\right)}{\left(\frac{z-2}{z+3}\right) + 1} = \frac{z-2}{(z-2)+(z+3)} = \frac{z-2}{2z+1}$$

- $T_1^{-1}T_2z$:
  First, find $T_1^{-1}(w)$ by solving $w = \dfrac{z-2}{z+3}$ for z:

$$w(z+3) = z-2 \implies wz + 3w = z - 2 \implies z(w-1) = -3w - 2$$

$$z = \frac{-3w-2}{w-1} = \frac{3w+2}{1-w} \implies T_1^{-1}(w) = \frac{3w+2}{1-w}$$

  Then:

$$T_1^{-1}(T_2(z)) = \frac{3\left(\dfrac{z}{z+1}\right)+2}{1-\left(\dfrac{z}{z+1}\right)} = \frac{3z+2(z+1)}{(z+1)-z} = \frac{5z+2}{1} = 5z+2$$

Q 29. Show that any linear transformation which transforms the real axis into itself can be written with real coefficients.

Solution: Let the Möbius transformation be defined by:

$$f(z) = \frac{az+b}{cz+d}, \quad ad - bc \neq 0$$

where $a, b, c, d \in \mathbb{C}$. We assume that $f(\mathbb{R} \cup \{\infty\}) = \mathbb{R} \cup \{\infty\}$.

Case 1: If $c = 0$, then $d \neq 0$ (otherwise $ad - bc = 0$). The transformation becomes:

$$f(z) = \frac{a}{d}z + \frac{b}{d}$$

Since $f(0) \in \mathbb{R}$, we must have $\dfrac{b}{d} \in \mathbb{R}$.
Since $f(1) \in \mathbb{R}$, we must have $\dfrac{a}{d} + \dfrac{b}{d} \in \mathbb{R}$, which implies $\dfrac{a}{d} \in \mathbb{R}$.
Let $k = d$. We can rewrite the coefficients as $a' = \dfrac{a}{k}$, $b' = \dfrac{b}{k}$, $c' = 0$, and $d' = 1$. All these are real.

Case 2: If $c \neq 0$, we can divide all coefficients by c to get:

$$f(z) = \frac{a'z+b'}{z+d'}$$

where $a' = \dfrac{a}{c}, b' = \dfrac{b}{c}, d' = \dfrac{d}{c}$.

- As $z \to \infty$, $f(z) \to a'$. Since the real axis maps to itself, a' must be real.

- The point z such that $f(z) = \infty$ is $z = -d'$. Since ∞ is on the extended real line, $-d'$ must be real, so d' is real.
- Finally, $f(0) = b'/d'$. Since b'/d' and d' are real, b' must also be real.

In both cases, there exists a complex number $\lambda \neq 0$ such that $\lambda a, \lambda b, \lambda c, \lambda d$ are all real. Thus, any such transformation can be written with real coefficients.

Q 30. Find the linear transformation which carries $0, i, -i$ into $1, -1, 0$.

Solution: The linear transformation $w = f(z)$ that maps $z_1 = 0, z_2 = i, z_3 = -i$ to $w_1 = 1, w_2 = -1, w_3 = 0$ is found using the cross-ratio formula:

$$\frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)} = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}$$

Substitute the given values into the formula:

$$\begin{aligned}
\implies \frac{(w-1)(-1-0)}{(w-0)(-1-1)} &= \frac{(z-0)(i-(-i))}{(z-(-i))(i-0)} \\
\implies \frac{-(w-1)}{-2w} &= \frac{z(2i)}{(z+i)(i)} \\
\implies \frac{w-1}{2w} &= \frac{2z}{z+i} \\
\implies (w-1)(z+i) &= 4wz \\
\implies wz + wi - z - i &= 4wz \\
\implies wi - 3wz &= z + i \\
\implies w(i - 3z) &= z + i
\end{aligned}$$

The required Möbius transformation is:

$$w = \frac{z + i}{i - 3z}$$

Q 31. Show that any four distinct points can be carried by a linear transformation to position $1, -1, k, -k$, where the value of k depends on the points.

Solution: To show that any four distinct points z_1, z_2, z_3, z_4 can be carried by a linear fractional transformation to the positions $1, -1, k, -k$, we use the fact that linear fractional transformations preserve the cross-ratio. The cross-ratio of four distinct points (z_1, z_2, z_3, z_4) is defined as:

$$\lambda = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}$$

Let the target points be $w_1 = 1, w_2 = -1, w_3 = k, w_4 = -k$. Then their cross-ratio is:

$$\lambda = \frac{(1-k)(-1-(-k))}{(1-(-k))(-1-k)} = \frac{(1-k)(k-1)}{(1+k)(-(1+k))} = \frac{-(1-k)^2}{-(1+k)^2} = \frac{(k-1)^2}{(k+1)^2}$$

For any given four distinct points, we calculate their cross-ratio λ. We then solve for k using the relation derived above:

$$\begin{aligned} \pm\sqrt{\lambda} &= \frac{k-1}{k+1} \\ \implies \pm\sqrt{\lambda}(k+1) &= k-1 \\ \implies k(\pm\sqrt{\lambda}-1) &= -1 \mp \sqrt{\lambda} \\ \implies k &= \frac{1 \pm \sqrt{\lambda}}{1 \mp \sqrt{\lambda}} \end{aligned}$$

Since the original points are distinct, $\lambda \neq 0, 1, \infty$. Therefore, we can always find a complex number k (specifically, $k \neq \pm 1, 0$) such that a transformation exists. Because the cross-ratio is the only invariant of four points under linear transformations, the existence of k guarantees the existence of the transformation.

Q 32. Find the most general linear transformation which carries the circle $|z| = 2$ into $|z+1| = 1$, the point -2 into the origin, and the origin into i.

Solution: The general linear transformation is given by:

$$f(z) = \frac{az+b}{cz+d}$$

Applying the given the conditions $f(-2) = 0$ and $f(0) = i$:

- $f(-2) = 0 \implies -2a + b = 0 \implies b = 2a$
- $f(0) = i \implies \dfrac{b}{d} = i \implies d = \dfrac{b}{i} = \dfrac{2a}{i} = -2ai$

Substituting these into the general form:

$$f(z) = \frac{az+2a}{cz-2ai} = \frac{a(z+2)}{cz-2ai}$$

Mapping the Circle $|z| = 2$ onto $|z+1| = 1$.
For any z where $|z| = 2$, the condition $|f(z)+1| = 1$ must hold:

$$\left| \frac{a(z+2) + (cz-2ai)}{cz-2ai} \right| = 1$$

$$|(a+c)z + 2a - 2ai| = |cz - 2ai|$$

Let $c = ka$. Factoring out $|a|$:

$$|(1+k)z + 2(1-i)| = |kz - 2i|$$

Setting $k = -(1+i)$ satisfies the symmetry and magnitude requirements for the mapping of the circles. Substituting $b = 2a$, $c = -(1+i)a$, and $d = -2ai$:

$$\begin{aligned} f(z) &= \frac{a(z+2)}{-a(1+i)z - 2ai} \\ &= \frac{z+2}{-(1+i)z - 2i} \\ &= \frac{(i-1)z + 2(i-1)}{-(1+i)z + 2} \end{aligned}$$

Q 33. Find the most general linear transformation of the circle $|z| = R$ into itself.

Solution: A Möbius transformation maps points symmetric with respect to a circle to points symmetric with respect to the image circle.

- Let α be a point inside the circle $|z| = R$ such that $f(\alpha) = 0$.
- The point symmetric to α with respect to $|z| = R$ is $z^* = \dfrac{R^2}{\overline{\alpha}}$.
- Since α maps to the center $w = 0$, its symmetric point z^* must map to the symmetric point of 0 with respect to $|z| = R$ (the image circle), which is $w = \infty$.

Because $f(\alpha) = 0$ and $f(R^2/\overline{\alpha}) = \infty$, the transformation must have the form:

$$w = k\frac{z-\alpha}{z - \dfrac{R^2}{\overline{\alpha}}} = K\frac{z-\alpha}{\overline{\alpha}z - R^2}$$

where K is a constant to be determined. Since the transformation maps $|z| = R$ to $|w| = R$, we test a point on the boundary, say $z = R$. We require $|w| = R$:

$$\left|K\frac{R-\alpha}{\overline{\alpha}R - R^2}\right| = R \implies \left|\frac{K}{R}\right| \cdot \left|\frac{R-\alpha}{\overline{\alpha} - R}\right| = R$$

Since $|R-\alpha| = |\overline{R-\alpha}| = |R-\overline{\alpha}|$, the fraction's magnitude is 1. Thus:

$$\left|\frac{K}{R}\right| = R \implies |K| = R^2$$

We can write $K = -R^2 e^{i\theta}$ to adjust the signs for the standard form. Substituting K back into the expression:

$$w = -R^2 e^{i\theta} \frac{z - \alpha}{\overline{\alpha} z - R^2}$$

Rearranging the negative sign into the denominator yields the standard general transformation:

$$w = Re^{i\theta} \frac{R(z - \alpha)}{R^2 - \overline{\alpha} z}$$

where $\theta \in \mathbb{R}$ and $|\alpha| < R$.

Q 34. A linear transformation with real coefficients carries a pair of concentric circles into another pair of concentric circles. Prove that the ratio of the radii must be the same.

Solution: Let C_1 and C_2 be two concentric circles in the complex plane with radii r_1 and r_2, respectively. Without loss of generality, assume they are centered at the origin $z = 0$. Let $f(z)$ be a linear transformation that carries these circles into another pair of concentric circles C_1' and C_2' with radii R_1 and R_2.
By the symmetry principle for the points at infinity, the point $z = 0$ (the center) and $z = \infty$ are symmetric with respect to both circles C_1 and C_2. Therefore, the images $f(0)$ and $f(\infty)$ must be symmetric with respect to both image circles C_1' and C_2'.
For two concentric circles in the finite plane, the only two points symmetric to both are their common center and the point at infinity. Thus, f must either map the center to the center and ∞ to ∞, or swap them.
Consider a line passing through the center of C_1 and C_2. It intersects the circles at four points. On the real axis, these points are $z_1 = r_1, z_2 = -r_1, z_3 = r_2, z_4 = -r_2$. The cross-ratio of these four points is defined as:

$$(z_1, z_2; z_3, z_4) = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}$$

Substituting the radius values:

$$(r_1, -r_1; r_2, -r_2) = \frac{(r_1 - r_2)(-r_1 + r_2)}{(r_1 + r_2)(-r_1 - r_2)} = \frac{-(r_1 - r_2)^2}{-(r_1 + r_2)^2} = \left(\frac{r_1 - r_2}{r_1 + r_2}\right)^2$$

Since Möbius transformations preserve the cross-ratio, the image points on C_1' and C_2' must satisfy:

$$\begin{aligned} \left(\frac{r_1 - r_2}{r_1 + r_2}\right)^2 &= \left(\frac{R_1 - R_2}{R_1 + R_2}\right)^2 \\ \implies \frac{r_1 - r_2}{r_1 + r_2} &= \pm \frac{R_1 - R_2}{R_1 + R_2} \end{aligned}$$

$$\implies (r_1 - r_2)(R_1 + R_2) = \pm(R_1 - R_2)(r_1 + r_2)$$

Simplifying either case (accounting for the possible swap of inner/outer circles) results in:

$$r_1 R_2 = r_2 R_1 \implies \frac{r_1}{r_2} = \frac{R_1}{R_2}$$

Thus, the ratio of the radii must be the same.

Q 35. Find a linear transformation which carries $|z| = 1$ and $|z - \frac{1}{4}| = \frac{1}{4}$ into concentric circles. What is the ratio of the radii?

Solution: To map two non-intersecting circles to concentric circles centered at the origin, we must find two points α and β that are symmetric with respect to both C_1 and C_2. If we map α to 0 and β to ∞, the resulting circles will be concentric about the origin.
Let α and β be these points. Since they lie on the line connecting the centers (the real axis), α and β are real.
For C_1 (center 0, radius 1), the symmetry condition is:

$$\alpha\beta = R^2 = 1 \implies \beta = \frac{1}{\alpha}$$

For C_2 (center $c = 1/4$, radius $r = 1/4$), the symmetry condition is:

$$(\alpha - c)(\beta - c) = r^2 \implies (\alpha - \frac{1}{4})(\frac{1}{\alpha} - \frac{1}{4}) = \frac{1}{16}$$

$$\implies 1 - \frac{\alpha}{4} - \frac{1}{4\alpha} + \frac{1}{16} = \frac{1}{16} \implies 1 - \frac{\alpha^2 + 1}{4\alpha} = 0 \implies 4\alpha = \alpha^2 + 1$$

Thus,

$$\alpha^2 - 4\alpha + 1 = 0$$

Using the quadratic formula:

$$\alpha = \frac{4 \pm \sqrt{16 - 4}}{2} = 2 \pm \sqrt{3}$$

Thus, our symmetric points are $\alpha = 2 - \sqrt{3}$ and $\beta = 2 + \sqrt{3}$.
We define the Möbius transformation $f(z)$ to send α to 0 and β to ∞:

$$f(z) = \frac{z - (2 - \sqrt{3})}{z - (2 + \sqrt{3})}$$

The images of the circles C_1 and C_2 will be circles centered at the origin with radii R_1 and R_2.
For C_1, we pick a point on the circle, $z = 1$:

$$R_1 = |f(1)| = \left|\frac{1-2+\sqrt{3}}{1-2-\sqrt{3}}\right| = \left|\frac{\sqrt{3}-1}{-(1+\sqrt{3})}\right| = \frac{\sqrt{3}-1}{\sqrt{3}+1} = \frac{4-2\sqrt{3}}{2} = 2-\sqrt{3}$$

For C_2, we pick the point $z = 0$ (which lies on C_2):

$$R_2 = |f(0)| = \left|\frac{0-(2-\sqrt{3})}{0-(2+\sqrt{3})}\right| = \frac{2-\sqrt{3}}{2+\sqrt{3}} = \frac{(2-\sqrt{3})^2}{4-3} = 7-4\sqrt{3}$$

The ratio of the radius of the inner circle to the outer circle is:

$$\frac{R_2}{R_1} = \frac{7-4\sqrt{3}}{2-\sqrt{3}} = \frac{(2-\sqrt{3})^2}{2-\sqrt{3}} = 2-\sqrt{3}$$

Q 36.

(a) Prove that (z_1, z_2, z_3, z_4) is real iff the four points lie on a circle or on a straight line.

(b) Prove the symmetry principle for linear transformations.

Solution:

(a) The cross ratio is defined as:

$$(z_1, z_2, z_3, z_4) = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}$$

$\Rightarrow$) Assume $(z_1, z_2, z_3, z_4) = \lambda$ where $\lambda \in \mathbb{R}$. Let $T(z)$ be the Möbius transformation that maps $z_2 \to 1$, $z_3 \to 0$, and $z_4 \to \infty$. By the property of Möbius transformations:

$$T(z_1) = (z_1, z_2, z_3, z_4) = \lambda$$

Since $\lambda \in \mathbb{R}$, the points $T(z_1), T(z_2), T(z_3), T(z_4)$ correspond to the points $\lambda, 1, 0, \infty$ on the extended real line $\mathbb{R} \cup \{\infty\}$. The real line is a generalized circle. Because the inverse Möbius transformation T^{-1} maps generalized circles to generalized circles, the original points z_1, z_2, z_3, z_4 must lie on a circle or a straight line.

$\Leftarrow$) Assume z_1, z_2, z_3, z_4 lie on a generalized circle C. There exists a Möbius transformation T that maps C onto the real line $\mathbb{R} \cup \{\infty\}$. Specifically, let $T(z_2) = 1, T(z_3) = 0, T(z_4) = \infty$. Since $z_1 \in C$, its image $T(z_1)$ must lie on the real line, so $T(z_1) \in \mathbb{R}$. Using the invariance of the cross ratio:

$$(z_1, z_2, z_3, z_4) = (T(z_1), T(z_2), T(z_3), T(z_4)) = (T(z_1), 1, 0, \infty) = T(z_1)$$

Since $T(z_1) \in \mathbb{R}$, the cross ratio is real.

(b) We prove that, if a Möbius transformation T maps a generalized circle C_1 onto C_2, then it maps any pair of points z, z^* symmetric with respect to C_1 to points $T(z), T(z^*)$ symmetric with respect to C_2.
Two points z and z^* are symmetric with respect to a generalized circle C passing through points z_1, z_2, z_3 if and only if:

$$\overline{(z, z_1, z_2, z_3)} = (z^*, z_1, z_2, z_3)$$

Let $w = T(z)$, $w^* = T(z^*)$, and $w_i = T(z_i)$ for $i = 1, 2, 3$. Since z_1, z_2, z_3 lie on C_1, their images w_1, w_2, w_3 lie on C_2.
By the invariance of the cross ratio under Möbius transformations:

$$(z, z_1, z_2, z_3) = (w, w_1, w_2, w_3) \quad \text{and} \quad (z^*, z_1, z_2, z_3) = (w^*, w_1, w_2, w_3)$$

Substituting these into the symmetry condition for z, z^*:

$$\overline{(w, w_1, w_2, w_3)} = (w^*, w_1, w_2, w_3)$$

This is exactly the condition for w and w^* to be symmetric with respect to the circle C_2 passing through w_1, w_2, w_3. Thus, $T(z)$ and $T(z^*)$ are symmetric with respect to C_2.

Q 37. Map the common part of the disks $|z| < 1$ and $|z - 1| < 1$ onto the inside of the unit circle.

Solution: The region is bounded by two circular arcs with common endpoints. The two common endpoints are

$$a = \frac{1}{2} + \frac{\sqrt{3}}{2}i, \quad b = \frac{1}{2} - \frac{\sqrt{3}}{2}i.$$

So, consider first the map

$$z_1 = \frac{z-a}{z-b} = \frac{z - \frac{1}{2} - \frac{\sqrt{3}}{2}i}{z - \frac{1}{2} + \frac{\sqrt{3}}{2}i} = \frac{2z - 1 - \sqrt{3}i}{2z - 1 + \sqrt{3}i}.$$

We know this maps the region Ω to a sector. To determine the boundary of this sector we make smart choices of z.
First consider $z = 1$, where we have

$$z_1(1) = \frac{1 - \sqrt{3}i}{1 + \sqrt{3}i} = e^{-i\pi/3 - i\pi/3} = e^{-2\pi i/3} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i.$$

Similarly, by choosing $z = 0$ we get

$$z_1(0) = \frac{a}{b} = e^{i\pi/3 - (-i\pi/3)} = e^{2\pi i/3} = \frac{1}{2} + \frac{\sqrt{3}}{2}i.$$

To determine the interior of the sector we make one more choice of z. Consider $z = \frac{1}{2}$,

$$z_1\left(\frac{1}{2}\right) = \frac{-\sqrt{3}i}{\sqrt{3}i} = -1.$$

It follows that z_1 maps Ω to the sector

$$\frac{2\pi}{3} \leq \theta \leq \frac{4\pi}{3}.$$

Consider now the map

$$z^{3/2} = e^{3/2 \log z}.$$

Exactly as we define $z^{\frac{1}{2}}$, we may consider the same branch cut by removing the negative real axis. But if we do that, we will remove a portion of $z_1(\Omega)$. To avoid this, we first rotate $z_1(\Omega)$. Let

$$z_2 = e^{i\pi/3} z_1.$$

Then z_2 rotates the sector $\frac{2\pi}{3} \leq \theta \leq \frac{4\pi}{3}$ to the sector

$$\pi \leq \theta \leq \frac{5\pi}{3}.$$

Now we can apply

$$z_3 = z_2^{3/2},$$

which expands this to the region

$$\frac{3\pi}{2} \leq \theta \leq \frac{5\pi}{2} \quad \Longleftrightarrow \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}.$$

This is the right half-plane. We know that the final transformation

$$w = \frac{z_3 - 1}{z_3 + 1}$$

maps this half-plane to the unit disk $|w| < 1$. The full composition is

$$w = \frac{z_3 - 1}{z_3 + 1} = \frac{z_2^{3/2} - 1}{z_2^{3/2} + 1} = \frac{(e^{i\pi/3} z_1)^{3/2} - 1}{(e^{i\pi/3} z_1)^{3/2} + 1} = \frac{i z_1^{3/2} - 1}{i z_1^{3/2} + 1}$$
$$= \frac{i(z - e^{i\pi/3})^{3/2}/(z - e^{-i\pi/3})^{3/2} - 1}{i(z - e^{i\pi/3})^{3/2}/(z - e^{-i\pi/3})^{3/2} + 1}.$$

Q 38. Map the region between $|z| = 1$ and $|z - \frac{1}{2}| = \frac{1}{2}$ onto a half–plane.

Solution: The region is bounded by two circles which are tangent at $a = 1$. The map

$$z_1 = \frac{1}{z - a} = \frac{1}{z - 1}$$

sends this region to a parallel strip. We test z_1 at different points to determine the boundary curves.
First consider $z = -1$ and $z = i$ which lie on $|z| = 1$. Then,

$$z_1(-1) = -\frac{1}{2}, \quad z_1(i) = \frac{1}{i - 1} = \frac{-1 - i}{2}.$$

This shows that one boundary curve is the line $x = -\frac{1}{2}$ in the z_1-plane.
Since the other boundary curve is parallel to this, we only need to test one point. Consider $z = 0$, which lies on the other circle. Then

$$z_1(0) = -1,$$

and so the other boundary curve is $x = -1$.
We can map this to the upper half-plane if the strip was bounded by $y = 0$ and $y = \pi$. This is easily done by a rotation, translation, and dilation.
First applying the rotation

$$z_2 = -iz_1$$

converts the strip between $y = \frac{1}{2}$ and $y = 1$.
Next, the translation

$$z_3 = z_2 - \frac{i}{2}$$

converts this to the strip between $y = 0$ and $y = \frac{1}{2}$.
Now, the dilation

$$z_4 = 2\pi z_3$$

converts this to the strip between $y = 0$ and $y = \pi$ as desired.
The final map

$$w = e^{z_4}$$

transforms this to the upper half-plane.
In total, the map is

$$\begin{aligned} w = e^{z_4} = e^{2\pi z_3} = e^{2\pi(z_2 - \frac{i}{2})} &= e^{2\pi z_2 - i\pi} \\ &= e^{-2\pi i z_1 - i\pi} = e^{-i\pi} e^{-2\pi i/(z-1)} \\ &= -e^{-2\pi i/(z-1)} \end{aligned}$$

Q 39. Map the complement of the arc $|z| = 1$, $\mathrm{Im}(z) > 0$ onto the outside of the unit circle so that the points at ∞ correspond to each other.

Solution: Consider the map

$$z_1 = \frac{z+1}{z-1}.$$

This maps the arc $|z| = 1, \mathrm{Im}(z) \geq 0$ to a straight line.
Choose $z = i$ so that

$$z_1(i) = \frac{1+i}{-1+i} = \frac{(1+i)(-1-i)}{2} = -i.$$

Since $z = i$ belongs to the specified arc, it follows that z_1 maps the arc to the ray $x = 0, \mathrm{Im}(z) \leq 0$. This map sends ∞ to 1.
Rotating the domain using

$$z_2 = -iz_1$$

takes the ray to the negative real axis, and the composition maps $\infty \mapsto -i$.
Using the standard branch cut for the square root, we use

$$z_3 = z_2^{1/2}$$

to map the region to the right half-plane. Moreover,

$$\infty \mapsto \sqrt{-i} = \left(e^{-\pi i/2}\right)^{1/2} = e^{-\pi i/4}.$$

Now translate and dilate so that $e^{-\frac{\pi i}{4}}$ is sent to 1. This is achieved via

$$z_4 = \sqrt{2}(z_3 + \frac{\sqrt{2}}{2}i) = \sqrt{2}z_3 + i.$$

Finally, we use the map

$$w = \frac{z_4+1}{z_4-1}$$

to map this to $|w| > 1$. It is clear that $\infty \mapsto \infty$.
The full composition is

$$
\begin{aligned}
w = \frac{z_4+1}{z_4-1} = \frac{\sqrt{2}z_3+i+1}{\sqrt{2}z_3+i-1} &= \frac{\sqrt{2}z_2^{1/2}+i+1}{\sqrt{2}z_2^{1/2}+i-1} \\
&= \frac{\sqrt{2}(-iz_1)^{1/2}+i+1}{\sqrt{2}(-iz_1)^{1/2}+i-1} \\
&= \frac{\sqrt{2}e^{-\pi i/4}z_1^{1/2}+i+1}{\sqrt{2}e^{-\pi i/4}z_1^{1/2}+i-1} \\
&= \frac{(1-i)z_1^{1/2}+i+1}{(1-i)z_1^{1/2}+i-1} \\
&= \frac{(1-i)\sqrt{\frac{z+1}{z-1}}+i+1}{(1-i)\sqrt{\frac{z+1}{z-1}}+i-1}
\end{aligned}
$$

To verify $w(\infty) = \infty$, define $\tilde{w}(z) = w(\frac{1}{z})$. Then,

$$\tilde{w}(z) = \frac{\sqrt{\frac{1+z}{1-z}}+i}{\sqrt{\frac{1+z}{1-z}}-i}.$$

Clearly $\tilde{w}(0) = \infty$, so $w(\infty) = \infty$.

Q 40. Find a conformal mapping of the disc $\mathbb{D}$ onto itself that takes $\dfrac{1}{2}$ to $\dfrac{1}{3}$.

Solution: Every automorphism of the unit disc $\mathbb{D}$ is a Möbius transformation of the form:

$$f(z) = e^{i\theta}\frac{z-a}{1-\bar{a}z}$$

where $|a| < 1$. For a real-valued mapping where $e^{i\theta} = 1$, we use:

$$f(z) = \frac{z-a}{1-az}$$

Then;

$$f(\frac{1}{2}) = \frac{1}{3} \implies \frac{\frac{1}{2}-a}{1-\frac{1}{2}a} = \frac{1}{3}$$

$$\implies 3\left(\frac{1}{2} - a\right) = 1 - \frac{1}{2}a \implies \frac{3}{2} - 3a = 1 - \frac{1}{2}a$$
$$\implies \frac{1}{2} = \frac{5}{2}a \implies a = \frac{1}{5}$$

Substituting $a = \frac{1}{5}$:

$$f(z) = \frac{z - \frac{1}{5}}{1 - \frac{1}{5}z} = \frac{5z-1}{5-z}$$

Q 41.

(a) Find a conformal equivalence of $\{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ with the open unit disc $\mathbb{D}$ such that $z = 1$ goes to $z = 0$.

(b) Find a conformal equivalence of $\{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ with itself such that $z = 1$ goes to $z = 2$.

Solution:

(a) A conformal map from the right half-plane $\mathbb{H} = \{z : \operatorname{Re}(z) > 0\}$ to the unit disc $\mathbb{D}$ is given by a Möbius transformation. We require $z = 1$ to map to $w = 0$, so the numerator must be $(z - 1)$. To ensure the boundary $\operatorname{Re}(z) = 0$ maps to $|w| = 1$, we use the standard transformation:

$$f(z) = \frac{z-1}{z+1}$$

Verification:

- **Point Map:** $f(1) = \frac{1-1}{1+1} = 0$.
- **Boundary:** For $z = iy$, $|f(iy)| = \left|\frac{iy-1}{iy+1}\right| = \frac{\sqrt{y^2+1}}{\sqrt{y^2+1}} = 1$.
- **Region:** Since $f(1) = 0 \in \mathbb{D}$, the right half-plane maps to the interior of the disc.

(b) An automorphism of the right half-plane can be constructed via a simple linear scaling. Since scaling by a positive real constant c preserves the condition $\operatorname{Re}(z) > 0$, we define:

$$g(z) = 2z$$

Verification:

- **Point Map:** $g(1) = 2(1) = 2$.
- **Preservation:** If $z = x + iy$ with $x > 0$, then $g(z) = 2x + 2iy$. Since $2x > 0$, the image remains in the right half-plane.
- **Bijectivity:** The map is clearly one-to-one and onto for the domain $\mathbb{H}$.

Q 42. Compute $\displaystyle\int_\gamma x\,dz$ where γ is the directed line segment from 0 to $1+i$.

Solution: The directed line segment from $z_1 = 0$ to $z_2 = 1 + i$ can be parametrized by:

$$z(t) = 0 + (1 + i - 0)t = (1 + i)t \quad , \quad 0 \le t \le 1$$

This gives the real and imaginary components:

$$z(t) = t + it \implies x(t) = t \quad , \quad y(t) = t$$

Differentiating $z(t)$ with respect to t:

$$dz = z'(t)\,dt = (1 + i)\,dt$$

Substitute the parametrization and the differential into the contour integral:

$$\begin{aligned}
\int_\gamma x\,dz &= \int_0^1 t \cdot (1 + i)\,dt \\
&= (1 + i) \int_0^1 t\,dt \\
&= (1 + i) \left[\frac{t^2}{2}\right]_0^1 \\
&= (1 + i) \left(\frac{1}{2} - 0\right) \\
&= \frac{1}{2} + \frac{i}{2}
\end{aligned}$$

Q 43. Compute $\displaystyle\int_{|z|=1} |z - 1|^2\,|dz|$.

Solution: The contour $|z| = 1$ represents the unit circle centered at the origin. We can parametrize this circle as:

$$z = e^{i\theta}, \quad 0 \le \theta \le 2\pi$$

To find the arc length element $|dz|$, we take the derivative:

$$dz = ie^{i\theta}\,d\theta \implies |dz| = |i||e^{i\theta}|\,d\theta = 1\,d\theta$$

Using the property $|w|^2 = w\overline{w}$ and the fact that $\overline{z} = \dfrac{1}{z}$ on the unit circle:

$$\begin{aligned}|z-1|^2 &= (z-1)(\overline{z}-1)\\ &= z\overline{z} - z - \overline{z} + 1\\ &= |z|^2 - (z+\overline{z}) + 1\end{aligned}$$

Substituting $|z|^2 = 1$ and the Euler identity $z + \overline{z} = 2\cos\theta$:

$$|z-1|^2 = 1 - 2\cos\theta + 1 = 2 - 2\cos\theta$$

Substitute the parametrization into the integral:

$$\begin{aligned}I &= \int_0^{2\pi} (2 - 2\cos\theta)\, d\theta\\ &= [2\theta - 2\sin\theta]_0^{2\pi}\\ &= (2(2\pi) - 2\sin(2\pi)) - (2(0) - 2\sin(0))\\ &= 4\pi - 0 - 0\\ &= 4\pi\end{aligned}$$

Q 44. Assume that $f(z)$ is analytic and satisfies $|f(z)-1| < 1$ in a domain Ω. Show that $\displaystyle\int_\gamma \frac{f'(z)}{f(z)}\, dz = 0$ for every closed curve in Ω.

Solution: The condition $|f(z)-1| < 1$ implies that for all $z \in \Omega$, the values of $f(z)$ lie within an open disk of radius 1 centered at $w = 1$. Let $\mathbb{D} = \{w \in \mathbb{C} : |w-1| < 1\}$. Since every $w \in \mathbb{D}$ satisfies $\mathrm{Re}(w) > 0$, the origin is not in $\mathbb{D}$ ($0 \notin \mathbb{D}$).
Because $f(\Omega) \subseteq \mathbb{D}$ and $\mathbb{D}$ is contained within the domain of the principal branch of the logarithm ($\mathbb{C} \setminus (-\infty, 0]$), we can define the function:

$$g(z) = \mathrm{Log}(f(z))$$

Since $f(z)$ is analytic and $f(z) \neq 0$ in Ω, $g(z)$ is a well-defined analytic function on Ω. By the chain rule, the derivative of $g(z)$ is:

$$g'(z) = \frac{d}{dz}\mathrm{Log}(f(z)) = \frac{1}{f(z)} \cdot f'(z) = \frac{f'(z)}{f(z)}$$

Thus, $\dfrac{f'(z)}{f(z)}$ has an analytic primitive $g(z)$ in Ω. By the Fundamental Theorem of Calculus for line integrals, if a function has a primitive, its integral over any closed curve γ (where $\gamma(a) = \gamma(b)$) is:

$$\int_\gamma \frac{f'(z)}{f(z)}\, dz = \int_\gamma g'(z)\, dz = g(\gamma(b)) - g(\gamma(a)) = 0$$

Q 45. If $P(z)$ is a polynomial and C denotes the circle $|z-a| = R$, evaluate $\displaystyle\int_C P(z)\,d\overline{z}$.

Solution: As $|z-a| = R$, then, $|z-a|^2 = R^2$.
Thus

$$\overline{(z-a)}(z-a) = R^2 \implies \overline{z} = \overline{a} + \frac{R^2}{z-a}$$

The differential $d\overline{z}$ is then:

$$d\overline{z} = -\frac{R^2}{(z-a)^2}\,dz$$

Then,

$$\mathrm{I} = \int_C P(z)\,d\overline{z} = -R^2 \int_C \frac{P(z)}{(z-a)^2}\,dz$$

Using Cauchy Residue Theorem,

$$\begin{aligned}
\mathrm{I} &= -R^2 \cdot 2\pi i \cdot \operatorname{Res}\left(\frac{P(z)}{(z-a)^2}, a\right) \\
&= -R^2 \cdot 2\pi i \cdot \left[\lim_{z\to a} \frac{d}{dz}\left((z-a)^2 \frac{P(z)}{(z-a)^2}\right)\right] \\
&= -R^2 \cdot 2\pi i \cdot \lim_{z\to a} P'(z) \\
&= -2\pi i R^2 \cdot P'(a)
\end{aligned}$$

Q 46. Determine explicitly the largest disk about the origin whose image under the mapping $w = z^2 + z$ is one-to-one.

Solution: To determine the largest disk about the origin $|z| < R$ where the mapping $w = f(z) = z^2 + z$ is injective, we analyze the conditions under which the function fails to be injective.
A function $f(z)$ is one-to-one on a domain if $f(z_1) = f(z_2)$ implies $z_1 = z_2$. For $f(z) = z^2 + z$, we set the images equal:

$$\begin{aligned}
& z_1^2 + z_1 && = z_2^2 + z_2 \\
\implies\ & z_1^2 - z_2^2 + z_1 - z_2 && = 0 \\
\implies\ & (z_1 - z_2)(z_1 + z_2) + (z_1 - z_2) && = 0 \\
\implies\ & (z_1 - z_2)(z_1 + z_2 + 1) && = 0
\end{aligned}$$

For distinct points ($z_1 \neq z_2$), the function fails to be one-to-one only if:

$$z_1 + z_2 = -1 \tag{1}$$

We find the largest radius R such that for all z_1, z_2 in the disk $|z| < R$, the equality $z_1 + z_2 = -1$ is impossible. By the triangle inequality:

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

If z_1 and z_2 are inside the disk $|z| < R$, then $|z_1| < R$ and $|z_2| < R$. This implies:

$$|z_1 + z_2| < R + R = 2R$$

To ensure $z_1 + z_2$ never equals -1, we must have:

$$2R \leq |-1| = 1 \implies R \leq \frac{1}{2}$$

A holomorphic function fails to be locally injective at its critical points. The derivative of $f(z)$ is:

$$f'(z) = 2z + 1$$

Setting $f'(z) = 0$ gives $z = -\frac{1}{2}$. This critical point lies exactly on the boundary of the disk $|z| = \frac{1}{2}$, confirming that any larger disk would contain points where the derivative vanishes or where distinct points map to the same image.

Q 47.

(a) Find an example of a non-constant entire function f such that $\sup_{x\in\mathbb{R}} |f(x)| < \infty$.

(b) Find an example of a non-constant entire function f such that $\sup_{x\in\mathbb{R}} |f(x)| + \sup_{y\in\mathbb{R}} |f(iy)| < \infty$.

Solution:

(a) Consider the function $f(z) = \sin(z)$.

- **Entirety:** The sine function is defined by the power series $\sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$, which converges for all $z \in \mathbb{C}$. Thus, $f(z)$ is entire.
- **Non-constant:** Since $f(0) = 0$ and $f(\frac{\pi}{2}) = 1$, the function is non-constant.
- **Boundedness on $\mathbb{R}$:** For any $x \in \mathbb{R}$, $|f(x)| = |\sin(x)| \leq 1$. Therefore:

$$\sup_{x\in\mathbb{R}} |f(x)| = 1 < \infty.$$

(b) Consider the function $f(z) = e^{-z^4}$.

- **Entirety:** $f(z)$ is the composition of the entire functions $g(z) = -z^4$ and $h(w) = e^w$. Therefore, $f(z)$ is entire.
- **Non-constant:** $f(0) = 1$ and $f(1) = e^{-1} \approx 0.368$, so f is non-constant.
- **Boundedness on $\mathbb{R}$:** For $z = x \in \mathbb{R}$:

$$|f(x)| = |e^{-x^4}| = e^{-x^4}.$$

Since $x^4 \geq 0$ for all real x, $0 < e^{-x^4} \leq 1$. Thus, $\sup_{x\in\mathbb{R}} |f(x)| = 1$.

- **Boundedness on $i\mathbb{R}$:** For $z = iy \in i\mathbb{R}$ (where $y \in \mathbb{R}$):

$$f(iy) = e^{-(iy)^4} = e^{-(i^4 y^4)} = e^{-(1\cdot y^4)} = e^{-y^4}.$$

As with the real axis, $|f(iy)| = e^{-y^4} \leq 1$. Thus, $\sup_{y\in\mathbb{R}} |f(iy)| = 1$.

- **Total Bound:**

$$\sup_{x\in\mathbb{R}} |f(x)| + \sup_{y\in\mathbb{R}} |f(iy)| = 1 + 1 = 2 < \infty.$$

Q 48. Prove that if $|a| \neq R$, then $\displaystyle\int_{|z|=R} \frac{|dz|}{|z-a||z+a|} < \frac{2\pi R}{|R^2 - |a|^2|}$.

Solution: We integrate both sides with respect to the arc length element $|dz|$ along the circle C defined by $|z| = R$:

$$\int_{|z|=R} \frac{|dz|}{|z-a||z+a|} < \int_{|z|=R} \frac{|dz|}{|R^2 - |a|^2|} = \frac{1}{|R^2 - |a|^2|} \int_{|z|=R} |dz|$$

The integral $\int_{|z|=R} |dz|$ represents the **arc length** (circumference) of the circle with radius R, which is $2\pi R$. Substituting this value, we obtain:

$$\int_{|z|=R} \frac{|dz|}{|z-a||z+a|} < \frac{2\pi R}{|R^2 - |a|^2|}$$

Q 49. Compute

(a) $\displaystyle\int_{|z|=1} \frac{e^z}{z}\, dz,$

(b) $\displaystyle\int_{|z|=2} \frac{dz}{z^2+1}.$

Solution:

(a) To compute this integral, we use **Cauchy's Integral Formula**, which states that if $g(z)$ is holomorphic inside and on a simple closed contour C, and z_0 is inside C, then:

$$\oint_C \frac{g(z)}{z - z_0}\,dz = 2\pi i \cdot g(z_0)$$

For the given integral:

- Let $g(z) = e^z$, which is **entire** (holomorphic everywhere in $\mathbb{C}$).
- The singularity occurs at $z_0 = 0$.
- The contour $|z| = 1$ is a circle of radius 1 centered at the origin, which encloses $z_0 = 0$.

Applying the formula:

$$\int_{|z|=1} \frac{e^z}{z}\,dz = 2\pi i \cdot e^0 = 2\pi i(1) = 2\pi i$$

(b) The integrand is $f(z) = \dfrac{1}{z^2+1}$. We can factor the denominator as $(z-i)(z+i)$. The singularities are at $z = i$ and $z = -i$. Both singularities lie inside the contour $|z| = 2$ since $|i| = 1 < 2$ and $|-i| = 1 < 2$.
We use the **Residue Theorem**:

$$\oint_C f(z)\,dz = 2\pi i \sum \operatorname{Res}(f, z_k)$$

The residue at $z = i$:

$$\operatorname{Res}(f, i) = \lim_{z\to i}(z-i)\frac{1}{(z-i)(z+i)} = \frac{1}{i+i} = \frac{1}{2i}$$

The residue at $z = -i$:

$$\operatorname{Res}(f, -i) = \lim_{z\to -i}(z+i)\frac{1}{(z-i)(z+i)} = \frac{1}{-i-i} = -\frac{1}{2i}$$

Therefore,

$$\int_{|z|=2} \frac{1}{z^2+1}\,dz = 2\pi i\left(\frac{1}{2i} + \left(-\frac{1}{2i}\right)\right) = 2\pi i(0) = 0$$

Q 50. Compute

(a) $\displaystyle\int_{|z|=1} e^z z^{-n}\, dz$,

(b) $\displaystyle\int_{|z|=2} z^n (1-z)^m\, dz$,

where $n, m = 0, \pm 1, \pm 2, \ldots$.

Solution:

(a) Let $f(z) = e^z z^{-n}$. The function $f(z)$ is analytic everywhere except at $z = 0$. The unit circle $|z| = 1$ encloses this singularity. Using the **Residue Theorem**:

$$\int_{|z|=1} f(z)\, dz = 2\pi i \cdot \text{Res}(f, 0)$$

We find the residue by expanding e^z into its Taylor series:

$$e^z z^{-n} = z^{-n} \sum_{k=0}^{\infty} \frac{z^k}{k!} = \sum_{k=0}^{\infty} \frac{z^{k-n}}{k!}$$

The residue at $z = 0$ is the coefficient of z^{-1}. Setting $k - n = -1$, we get $k = n - 1$.
Case Analysis:

- If $n \le 0$: The power $k - n$ is always non-negative ($k \ge 0$ and $-n \ge 0$). There is no z^{-1} term, so $\text{Res}(f, 0) = 0$.
- If $n \ge 1$: The coefficient at $k = n - 1$ is $\dfrac{1}{(n-1)!}$.

Therefore,

$$\int_{|z|=1} e^z z^{-n}\, dz = \begin{cases} \dfrac{2\pi i}{(n-1)!} & \text{if } n \ge 1 \\ 0 & \text{if } n \le 0 \end{cases}$$

(b) The function $f(z) = z^n (1-z)^m$ has singularities only at $z = 0$ and $z = 1$, both of which are inside the contour $|z| = 2$. It is most efficient to use the **Residue at Infinity**:

$$\int_{|z|=2} f(z)\, dz = -2\pi i \cdot \text{Res}(f, \infty)$$

The residue at infinity is defined as $\text{Res}(f, \infty) = \text{Res}\left(-\frac{1}{w^2} f\left(\frac{1}{w}\right), 0\right)$.

$$f\left(\frac{1}{w}\right) = \left(\frac{1}{w}\right)^n \left(1 - \frac{1}{w}\right)^m = w^{-n} \left(\frac{w-1}{w}\right)^m = w^{-(n+m)} (w-1)^m$$

$$-\frac{1}{w^2} f\left(\frac{1}{w}\right) = -w^{-(n+m+2)}(w-1)^m = -w^{-(n+m+2)} \sum_{k=0}^{m} \binom{m}{k} w^k (-1)^{m-k}$$

We know that, if $m < 0$, the generalized binomial expansion $(1-z)^m = \sum \binom{m}{k}(-z)^k$.
The residue is the coefficient of w^{-1} in this expansion. This occurs when:

$$-(n+m+2)+k = -1 \implies k = n+m+1$$

Case Analysis:

- If $n+m+1 < 0$: There is no such $k \geq 0$ in the expansion, so the residue is 0.
- If $n+m+1 \geq 0$: The coefficient (residue) is $-\binom{m}{n+m+1}(-1)^{m-(n+m+1)} = -\binom{m}{n+m+1}(-1)^{-n-1}$.

Therefore,

$$\int_{|z|=2} z^n(1-z)^m \, dz = 2\pi i \binom{m}{n+m+1}(-1)^{-n-1}$$

The integral is non-zero only if $n+m \geq -1$. Specifically, if $n, m \geq 0$, the function is a polynomial and the integral is 0.

Q 51. Prove that a function analytic in the whole plane and satisfying $|f(z)| < |z|^n$ for some $n > 0$ and all sufficiently large $|z|$ reduces to a polynomial.

Solution: Since $f(z)$ is entire, it can be represented by a power series centered at the origin that converges for all $z \in \mathbb{C}$:

$$f(z) = \sum_{k=0}^{\infty} a_k z^k$$

By **Cauchy's Inequality**, for any circle $|z| = R$, the coefficients a_k satisfy:

$$|a_k| \leq \frac{M(R)}{R^k}$$

where $M(R) = \max_{|z|=R} |f(z)|$. From the given growth condition, for sufficiently large R, we have $M(R) < R^n$. Substituting this into the inequality for the k-th coefficient:

$$|a_k| \leq \frac{R^n}{R^k} = R^{n-k}$$

Now, consider any index $k > n$. As we let the radius $R \to \infty$, the term R^{n-k} approaches zero because the exponent is negative:

$$\lim_{R\to\infty} |a_k| \leq \lim_{R\to\infty} R^{n-k} = 0$$

This implies that $a_k = 0$ for all integers $k > n$. Consequently, the Taylor series for $f(z)$ terminates:

$$f(z) = \sum_{k=0}^{n} a_k z^k$$

Thus, $f(z)$ is a polynomial of degree at most n.

Q 52. Show that the successive derivatives of an analytic function at a point can never satisfy $|f^{(n)}(z)| > n!\, n^n$.

Solution: To show that the successive derivatives of a function $f(z)$ analytic at z_0 cannot satisfy $|f^{(n)}(z_0)| > n!\, n^n$ for all n, we examine the convergence of its Taylor series. If $f(z)$ is analytic at z_0, it can be represented by a power series in a neighborhood of z_0:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad \text{where } a_n = \frac{f^{(n)}(z_0)}{n!}$$

The radius of convergence R for this power series is determined by the Cauchy-Hadamard formula:

$$\frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{|a_n|} = \limsup_{n\to\infty} \left(\frac{|f^{(n)}(z_0)|}{n!} \right)^{1/n}$$

Assume that $|f^{(n)}(z_0)| > n!\, n^n$ for all $n \in \mathbb{N}$. Substituting this inequality into the limit:

$$\frac{1}{R} \geq \limsup_{n\to\infty} \left(\frac{n!\, n^n}{n!} \right)^{1/n} = \limsup_{n\to\infty} (n^n)^{1/n} = \limsup_{n\to\infty} n = \infty$$

If $\dfrac{1}{R} = \infty$, then $R = 0$. However, by definition, a function analytic at a point must have a positive radius of convergence, i.e., $R > 0$. This contradiction proves that the growth of the derivatives cannot exceed $n!\, n^n$ for all n.

Q 53. Suppose that f is analytic on a domain $G \subset \mathbb{C}$ and $\{z : |z - a| \leq R\} \subset G$. If $|f(z)| \leq M$ for $|z - a| = R$, show that for any $w_1, w_2 \in \{w : |w - a| \leq \frac{1}{2}R\}$,

$$|f(w_1) - f(w_2)| \leq \frac{4M}{R} |w_1 - w_2|.$$

Solution: To prove this, we first bound the derivative $f'(z)$ for any z inside the disk $|z - a| \leq \dfrac{1}{2}R$. By the Cauchy Integral Formula for the derivative, for any z such that $|z - a| \leq \dfrac{1}{2}R$, we have:

$$f'(z) = \frac{1}{2\pi i} \oint_{|\zeta - a| = R} \frac{f(\zeta)}{(\zeta - z)^2}\, d\zeta$$

Using the ML-Inequality, where $L = 2\pi R$ is the length of the circle and M is the bound on $|f|$:

$$|f'(z)| \leq \frac{1}{2\pi} \cdot \max_{|\zeta - a| = R} \left| \frac{f(\zeta)}{(\zeta - z)^2} \right| \cdot 2\pi R = \frac{MR}{\min\limits_{|\zeta - a| = R} |\zeta - z|^2}$$

Since $|\zeta - a| = R$ and $|z - a| \leq \frac{1}{2}R$, by the Reverse Triangle Inequality:

$$|\zeta - z| = |(\zeta - a) - (z - a)| \geq |\zeta - a| - |z - a| \geq R - \frac{1}{2}R = \frac{1}{2}R$$

Therefore, $|\zeta - z|^2 \geq \frac{1}{4}R^2$. Substituting this back into the derivative bound:

$$|f'(z)| \leq \frac{MR}{\frac{1}{4}R^2} = \frac{4M}{R}$$

For any w_1, w_2 in the convex disk $\{w : |w - a| \leq \frac{1}{2}R\}$, the line segment connecting them lies entirely within the region where our bound for $f'(z)$ holds. We can write:

$$f(w_1) - f(w_2) = \int_{w_2}^{w_1} f'(z)\, dz \implies |f(w_1) - f(w_2)| \leq \max_{z \in [w_2, w_1]} |f'(z)| \cdot |w_1 - w_2|$$

Applying our previous result $|f'(z)| \leq \frac{4M}{R}$:

$$|f(w_1) - f(w_2)| \leq \frac{4M}{R} |w_1 - w_2|$$

Q 54. Show that a function analytic in the whole plane and having a nonessential singularity at ∞ reduces to a polynomial.

Solution: Let $f(z)$ be an entire function. Suppose $f(z)$ has a **nonessential singularity** at ∞. By definition, the behavior of $f(z)$ at ∞ is determined by the behavior of $g(w) = f(\frac{1}{w})$ at $w = 0$. A nonessential singularity means $g(w)$ has either a **removable singularity** or a **pole** at $w = 0$.
Case Analysis:

- If $g(w)$ has a removable singularity at $w = 0$, then $\lim_{w \to 0} g(w)$ exists and is finite. This implies:

$$\lim_{z \to \infty} f(z) = L < \infty$$

  Since $f(z)$ is entire and bounded as $z \to \infty$, it is bounded on the entire complex plane. By **Liouville's Theorem**, $f(z)$ must be a constant, which is a polynomial of degree 0.

Continued...

- If $g(w)$ has a pole of order m at $w = 0$, its Laurent series near $w = 0$ is:

$$g(w) = \sum_{n=-m}^{\infty} a_n w^n = \frac{a_{-m}}{w^m} + \dots + \frac{a_{-1}}{w} + a_0 + a_1 w + \dots$$

Substituting $w = \frac{1}{z}$, we find the expansion of $f(z)$ for large $|z|$:

$$f(z) = a_{-m} z^m + a_{-m+1} z^{m-1} + \dots + a_{-1} z + a_0 + \frac{a_1}{z} + \dots$$

Since $f(z)$ is entire, its power series expansion $f(z) = \sum_{n=0}^{\infty} c_n z^n$ must match this Laurent expansion in the region where both are valid. This forces all coefficients $c_n = 0$ for $n > m$ and all coefficients $a_n = 0$ for $n > 0$.

In either case, $f(z)$ takes the form:

$$f(z) = \sum_{n=0}^{m} c_n z^n$$

which is a **polynomial** of degree m.

Q 55. Show that the functions e^z, $\sin z$, and $\cos z$ have essential singularities at ∞.

Solution: To show that a function $f(z)$ has an essential singularity at $z = \infty$, we examine the behavior of $g(w) = f(\frac{1}{w})$ at $w = 0$.

- The Laurent series for e^z about $z = 0$ is:

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \dots$$

Replacing z with $\frac{1}{w}$:

$$e^{1/w} = \sum_{n=0}^{\infty} \frac{1}{n! w^n} = 1 + \frac{1}{w} + \frac{1}{2! w^2} + \frac{1}{3! w^3} + \dots$$

Since the principal part (the terms with negative powers of w) has infinitely many non-zero terms, $w = 0$ is an **essential singularity** for $e^{1/w}$. Thus, $z = \infty$ is an essential singularity for e^z.

- The Laurent series for $\sin z$ is:

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$$

Replacing z with $\frac{1}{w}$:

$$\sin(\frac{1}{w}) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!w^{2n+1}} = \frac{1}{w} - \frac{1}{3!w^3} + \frac{1}{5!w^5} - \dots$$

The expansion contains infinitely many negative powers of w, confirming an **essential singularity** at $z = \infty$.

- For $\cos z$:

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$$

Replacing z with $\frac{1}{w}$:

$$\cos(\frac{1}{w}) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!w^{2n}} = 1 - \frac{1}{2!w^2} + \frac{1}{4!w^4} - \dots$$

As with the previous functions, the infinite number of negative power terms in w indicates an **essential singularity** at $z = \infty$.

Q 56. Prove using Schwarz's lemma that every one-to-one conformal mapping of a circular disc onto another circular disc (or a half-plane) is given by a linear transformation.

Solution: Let $f : \mathbb{D} \to \mathbb{D}$ be a conformal automorphism of the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Let $\alpha = f^{-1}(0) \in \mathbb{D}$. Define the **Möbius transformation** (automorphism):

$$\psi(z) = \frac{z - \alpha}{1 - \overline{\alpha}z}$$

Since $\psi(\alpha) = 0$, the composition $g = f \circ \psi^{-1}$ is a conformal map from $\mathbb{D}$ to $\mathbb{D}$ such that $g(0) = 0$. By **Schwarz's Lemma**, because $g : \mathbb{D} \to \mathbb{D}$ is holomorphic and $g(0) = 0$, we have:

$$|g(z)| \leq |z| \quad \text{for all } z \in \mathbb{D}$$

Since f is a bijection, g is also a bijection. Applying the same logic to the inverse map $g^{-1} : \mathbb{D} \to \mathbb{D}$, where $g^{-1}(0) = 0$:

$$|g^{-1}(w)| \leq |w|$$

Setting $w = g(z)$, we obtain $|z| \leq |g(z)|$. Since $|g(z)| \leq |z|$ and $|g(z)| \geq |z|$, it follows that $|g(z)| = |z|$ for all $z \in \mathbb{D}$. By the equality case of Schwarz's Lemma, g must be a rotation:

$$g(z) = e^{i\theta} z$$

Thus,

$$f(z) = g(\psi(z)) = e^{i\theta}\frac{z-\alpha}{1-\overline{\alpha}z}$$

Because f is the composition of two Möbius transformations, it is itself a **Möbius transformation**. Any circular disc or half-plane can be mapped to the unit disc $\mathbb{D}$ via a specific Möbius transformation (e.g., the **Cayley transform** for half-planes). Therefore, any conformal map between such domains is a composition of Möbius transformations, which remains linear fractional.

Q 57. Let f be analytic in $|z| < 1$. Show that $|f(z)| \leq 1$ for $|z| < 1$ implies

$$\frac{|f'(z)|}{1-|f(z)|^2} \leq \frac{1}{1-|z|^2}, \qquad (|z| < 1).$$

Solution: Fix an arbitrary point $z_0 \in \mathbb{D}$. Let $w_0 = f(z_0)$. We define two Möbius transformations that are automorphisms of the unit disk:

$$\psi(z) = \frac{z-z_0}{1-\overline{z_0}z} \quad \text{and} \quad \phi(w) = \frac{w-w_0}{1-\overline{w_0}w}$$

Now, consider the composition $g = \phi \circ f \circ \psi^{-1}$. Note that:

- g maps $\mathbb{D}$ to $\mathbb{D}$ because f, ϕ, and ψ^{-1} all map the disk to itself.
- $g(0) = \phi(f(z_0)) = \phi(w_0) = 0$.

By **Schwarz's Lemma**, any analytic function $g : \mathbb{D} \to \mathbb{D}$ with $g(0) = 0$ satisfies $|g'(0)| \leq 1$. We calculate $g'(0)$ using the Chain Rule:

$$g'(0) = \phi'(f(z_0)) \cdot f'(z_0) \cdot (\psi^{-1})'(0)$$

The derivative of a Möbius transformation $\psi(z) = \dfrac{z-a}{1-\bar{a}z}$ is $\psi'(z) = \dfrac{1-|a|^2}{(1-\bar{a}z)^2}$ Then,

$$\phi'(w_0) = \frac{1}{1-|w_0|^2} \quad \text{and} \quad (\psi^{-1})'(0) = 1-|z_0|^2$$

Substituting these into the inequality $|g'(0)| \leq 1$:

$$\left|\frac{1}{1-|f(z_0)|^2} \cdot f'(z_0) \cdot (1-|z_0|^2)\right| \leq 1$$

Rearranging terms yields the desired inequality:

$$\frac{|f'(z_0)|}{1-|f(z_0)|^2} \leq \frac{1}{1-|z_0|^2}$$

Continued...

Since z_0 was arbitrary, the result holds for all $z \in \mathbb{D}$.

Q 58. Suppose that $f(\zeta)$ is continuous on the arc γ. Prove that $F_n(z) = \displaystyle\int_\gamma \frac{f(\zeta)\,d\zeta}{(\zeta - z)^n}$, $n = 1, 2, 3, \dots$, is analytic in each of the regions determined by γ, and that $F_n'(z) = nF_{n+1}(z)$, where $n = 2, 3, \dots$.

Solution: Suppose that $f(\zeta)$ is continuous on the arc γ. We define the function:

$$F_n(z) = \int_\gamma \frac{f(\zeta)\,d\zeta}{(\zeta - z)^n}, \quad n = 1, 2, 3, \dots$$

for $z \notin \gamma$. To prove that $F_n(z)$ is analytic in each region determined by γ, we show that it is complex-differentiable at any point z not on the arc. Let D be a region not containing γ. For any $z \in D$, the distance $d = \text{dist}(z, \gamma) > 0$. The integrand $g(z, \zeta) = \dfrac{f(\zeta)}{(\zeta - z)^n}$ is a continuous function of both variables for $\zeta \in \gamma$ and $z \in D$. Furthermore, the partial derivative

$$\frac{\partial g}{\partial z} = \frac{nf(\zeta)}{(\zeta - z)^{n+1}}$$

is also continuous. According to the theorem for **differentiation under the integral sign**, if the integrand and its partial derivative are continuous, then $F_n(z)$ is holomorphic (analytic) in D. We compute the derivative $F_n'(z)$ by applying the operator inside the integral:

$$F_n'(z) = \frac{d}{dz}\int_\gamma \frac{f(\zeta)}{(\zeta - z)^n}\,d\zeta = \int_\gamma \frac{\partial}{\partial z}\left[f(\zeta)(\zeta - z)^{-n}\right]\,d\zeta$$

Using the chain rule for the term $(\zeta - z)^{-n}$:

$$\frac{\partial}{\partial z}(\zeta - z)^{-n} = -n(\zeta - z)^{-n-1} \cdot \frac{d}{dz}(\zeta - z) = -n(\zeta - z)^{-n-1} \cdot (-1) = \frac{n}{(\zeta - z)^{n+1}}$$

Substituting this result back into the integral:

$$F_n'(z) = \int_\gamma \frac{nf(\zeta)}{(\zeta - z)^{n+1}}\,d\zeta = n\int_\gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}}\,d\zeta = nF_{n+1}(z)$$

This holds for $n = 1, 2, 3, \dots$, and since F_n is differentiable, it is analytic in the complement of γ.

Q 59. Let f be analytic in $H = \{z = x + iy : -\infty < x < \infty,\ y > 0\}$. Suppose there exists $\delta > 0$ such that for every real x with $|x| < \delta$, $\lim\limits_{\substack{z \to x \\ z \in H}} f(z) = 0$. Prove that $f \equiv 0$ on H.

Solution: To prove that $f \equiv 0$, we utilize the **Schwarz Reflection Principle** and the **Identity Theorem**. Let $I = (-\delta, \delta)$ be the open interval on the real axis where f vanishes in the limit. We define a new function $F(z)$ on the domain $D = H \cup I \cup H^*$, where $H^* = \{z \in \mathbb{C} : \overline{z} \in H\}$ is the lower half-plane:

$$F(z) = \begin{cases} f(z) & z \in H \\ 0 & z \in I \\ \overline{f(\overline{z})} & z \in H^* \end{cases}$$

Using $\overline{f(\overline{z})}$ ensures the extension is analytic. By the **Schwarz Reflection Principle**, since f is analytic in H and approaches a real-valued constant (zero) continuously on the boundary segment I, the function $F(z)$ is analytic on the union D. Specifically, the continuity of F across the real segment I is guaranteed by the given limit $\lim_{z \to x} f(z) = 0$. We observe the following:

- $F(z)$ is analytic on the connected domain D.
- $F(z) = 0$ for all $z \in I$.
- The interval $I = (-\delta, \delta)$ is a set that contains limit points within D.

According to the **Identity Theorem**, if an analytic function on a connected domain vanishes on a set containing a limit point, it must be identically zero on the entire domain. Thus, $F(z) \equiv 0$ for all $z \in D$.
Therefore, $f \equiv 0$ for all $z \in H$.

Q 60. Map the region $\{z = x + iy : -1 < x < 1,\ y > 0\}$ onto $H = \{w = u + iv : -\infty < u < \infty,\ v > 0\}$.

Solution: The standard conformal map using the sine function takes a strip of width π (from $-\pi/2$ to $\pi/2$) to the upper half-plane. Since our strip has a width of 2 (from -1 to 1), we first scale the variable z:

$$\zeta = \frac{\pi}{2} z$$

This maps our region to the strip $-\frac{\pi}{2} < \text{Re}(\zeta) < \frac{\pi}{2}$ with $\text{Im}(\zeta) > 0$. The sine function $w = \sin(\zeta)$ maps this normalized strip onto the upper half-plane H. Substituting the expression for ζ gives the final transformation:

$$w(z) = \sin\left(\frac{\pi z}{2}\right)$$

- **Vertical Boundary ($x = 1, y > 0$):** $w = \sin\left(\frac{\pi}{2} + i\frac{\pi y}{2}\right) = \cos(i\frac{\pi y}{2}) = \cosh(\frac{\pi y}{2})$. This maps to the real interval $(1, \infty)$.

- **Vertical Boundary ($x = -1, y > 0$):** $w = \sin\left(-\frac{\pi}{2} + i\frac{\pi y}{2}\right) = -\cosh(\frac{\pi y}{2})$. This maps to the real interval $(-\infty, -1)$.
- **Horizontal Boundary ($y = 0, -1 < x < 1$):** $w = \sin\left(\frac{\pi x}{2}\right)$. This maps to the real interval $(-1, 1)$.

Q 61. Let $f : D \to D$ be analytic, where D is the open unit disc. Suppose $f(1/4) = -2/3$. Is it possible that $f(1/3) = 2/3$? Explain your reasoning.

Solution: According to the **Schwarz-Pick Theorem**, any analytic function mapping the open unit disc D to itself must satisfy the following inequality for any $z_1, z_2 \in D$:

$$\left|\frac{f(z_1) - f(z_2)}{1 - \overline{f(z_1)}f(z_2)}\right| \leq \left|\frac{z_1 - z_2}{1 - \overline{z_1}z_2}\right|$$

Let $z_1 = \frac{1}{4}$ and $z_2 = \frac{1}{3}$. We are given $f(z_1) = -\frac{2}{3}$ and $f(z_2) = \frac{2}{3}$. Then:

$$\text{RHS} = \left|\frac{\frac{1}{4} - \frac{1}{3}}{1 - (\frac{1}{4})(\frac{1}{3})}\right| = \left|\frac{-\frac{1}{12}}{\frac{11}{12}}\right| = \frac{1}{11} \approx 0.0909$$

And

$$\text{LHS} = \left|\frac{-\frac{2}{3} - \frac{2}{3}}{1 - (-\frac{2}{3})(\frac{2}{3})}\right| = \left|\frac{-\frac{4}{3}}{1 + \frac{4}{9}}\right| = \left|\frac{-\frac{4}{3}}{\frac{13}{9}}\right| = \frac{4}{3} \cdot \frac{9}{13} = \frac{12}{13} \approx 0.9231$$

The Schwarz-Pick inequality requires:

$$\frac{12}{13} \leq \frac{1}{11}$$

Since $0.9231 \not\leq 0.0909$, the condition is violated. Therefore, no such analytic function exists.

Q 62. If f is analytic and $\operatorname{Im} f(z) > 0$ for $\operatorname{Im} z > 0$, show that

$$\left|\frac{f(z) - f(z_0)}{f(z) - \overline{f(z_0)}}\right| \leq \left|\frac{z - z_0}{z - \overline{z_0}}\right|.$$

Solution: Let $\mathbb{H} = \{w \in \mathbb{C} : \operatorname{Im}(w) > 0\}$ denote the upper half-plane. The given condition states that f is an analytic map such that $f : \mathbb{H} \to \mathbb{H}$.

To prove the inequality, we utilize the Schwarz-Pick Lemma by transforming the upper half-plane into the unit disk $\mathbb{D} = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$.
Consider the Cayley transform $\psi_\alpha : \mathbb{H} \to \mathbb{D}$ defined by:

$$\psi_\alpha(w) = \frac{w - \alpha}{w - \overline{\alpha}}, \quad \text{for } \alpha \in \mathbb{H}$$

Now, fix $z_0 \in \mathbb{H}$ and define a function $g : \mathbb{D} \to \mathbb{D}$ as the composition:

$$g = \psi_{f(z_0)} \circ f \circ \psi_{z_0}^{-1}$$

We observe two properties of g:

- g is analytic on $\mathbb{D}$ because it is a composition of analytic functions.
- $g(0) = \psi_{f(z_0)}(f(\psi_{z_0}^{-1}(0))) = \psi_{f(z_0)}(f(z_0)) = \dfrac{f(z_0) - f(z_0)}{f(z_0) - \overline{f(z_0)}} = 0.$

By Schwarz's Lemma, any analytic function $g : \mathbb{D} \to \mathbb{D}$ fixing the origin satisfies $|g(\zeta)| \le |\zeta|$ for all $\zeta \in \mathbb{D}$. Setting $\zeta = \psi_{z_0}(z)$, we have:

$$|g(\psi_{z_0}(z))| \le |\psi_{z_0}(z)|$$

Substituting the definition of g:

$$|\psi_{f(z_0)}(f(z))| \le |\psi_{z_0}(z)|$$

Expanding the terms using the definition of ψ, we obtain the desired inequality:

$$\left| \frac{f(z) - f(z_0)}{f(z) - \overline{f(z_0)}} \right| \le \left| \frac{z - z_0}{z - \overline{z_0}} \right|$$

Q 63. Find the residues of the given functions at all their isolated singularities in $\mathbb{C}$.

(a) $\dfrac{1}{z^3 - z^5}$

(b) $\dfrac{z^2}{(z^2 + 1)^2}$

(c) $\dfrac{e^z}{z^2(z^2 + 9)}$

Solution:

a) **Function:** $f(z) = \dfrac{1}{z^3(1-z^2)}$

- **Singularities:** $z = 0$ (pole of order 3) and $z = \pm 1$ (simple pole).
- **Residue at** $z = 0$**:**

$$\text{Res }(f,0) = \frac{1}{(3-1)!}\lim_{z\to 0}\frac{d^2}{dz^2}\left[\frac{1}{1-z^2}\right] = \frac{1}{2}\lim_{z\to 0}\frac{8z+2(1-z^2)}{(1-z^2)^3} = 1$$

- **Residue at** $z = 1$**:**

$$\text{Res }(f,1) = \lim_{z\to 1}(z-1)\frac{1}{z^3(1-z)(1+z)} = \frac{-1}{(1)^3(1+1)} = \frac{-1}{2}$$

- **Residue at** $z = -1$**:**

$$\text{Res }(f,-1) = \lim_{z\to -1}(z+1)\frac{1}{z^3(1-z)(1+z)} = \frac{1}{(-1)^3(1-(-1))} = \frac{-1}{2}$$

b) **Function:** $f(z) = \dfrac{z^2}{(z^2+1)^2} = \dfrac{z^2}{(z-i)^2(z+i)^2}$

- **Singularities:** $z = i$ and $z = -i$ (both poles of order 2).
- **Residue at** $z = i$**:**

$$\begin{aligned}\text{Res }(f,i) = \lim_{z\to i}\frac{d}{dz}\left[\frac{z^2}{(z+i)^2}\right] &= \lim_{z\to i}\frac{2z(z+i)-2z^2}{(z+i)^3}\\ &= \frac{2i(2i)-2(i)^2}{(2i)^3}\\ &= \frac{-4+2}{-8i} = \frac{-2}{-8i} = -\frac{i}{4}\end{aligned}$$

- **Residue at** $z = -i$**:**

$$\begin{aligned}\text{Res }(f,-i) = \lim_{z\to -i}\frac{d}{dz}\left[\frac{z^2}{(z-i)^2}\right] &= \frac{2(-i)(-2i)-2(-i)^2}{(-2i)^3}\\ &= \frac{-4+2}{8i} = \frac{-2}{8i} = \frac{i}{4}\end{aligned}$$

c) **Function:** $f(z) = \dfrac{e^z}{z^2(z^2+9)} = \dfrac{e^z}{z^2(z-3i)(z+3i)}$

- **Singularities:** $z = 0$ (pole of order 2), $z = 3i$ and $z = -3i$ (simple poles).
- **Residue at** $z = 0$**:**

$$\text{Res }(f,0) = \lim_{z\to 0}\frac{d}{dz}\left[\frac{e^z}{z^2+9}\right] = \lim_{z\to 0}\frac{e^z(z^2+9)-2ze^z}{(z^2+9)^2} = \frac{1(9)-0}{81} = \frac{1}{9}$$

- **Residue at $z = 3i$:**

$$\text{Res }(f, 3i) = \lim_{z\to 3i}(z-3i)\frac{e^z}{z^2(z-3i)(z+3i)} = \frac{e^{3i}}{(3i)^2(6i)} = \frac{e^{3i}}{-9(6i)} = \frac{ie^{3i}}{54}$$

- **Residue at $z = -3i$:**

$$\begin{aligned}\text{Res }(f, -3i) &= \lim_{z\to -3i}(z+3i)\frac{e^z}{z^2(z-3i)(z+3i)}\\ &= \frac{e^{-3i}}{(-3i)^2(-6i)}\\ &= \frac{e^{-3i}}{-9(-6i)}\\ &= -\frac{ie^{-3i}}{54}\end{aligned}$$

Q 64. Evaluate the integrals, assuming that the closed contours are traversed in the positive direction.

(a) $\displaystyle\int_C \frac{dz}{z^4+1}$, where C is the circle $x^2+y^2=2x$

(b) $\displaystyle\int_C \frac{zdz}{z^4+1}$, where C is the circle $|z-2| = \frac{1}{2}$

(c) $\displaystyle\int_C \frac{z^3dz}{2z^4+1}$, where C is the circle $|z| = 1$

Solution:

(a) The contour $x^2+y^2=2x$ can be rewritten as $(x-1)^2+y^2=1$, which is a circle centered at 1 with radius 1. The singularities of $f(z) = \dfrac{1}{z^4+1}$ are the roots of $z^4=-1$, given by:

$$z_k = e^{i(\pi+2k\pi)/4} \quad , \quad k = 0, 1, 2, 3$$

The poles are:

$$z_0 = \frac{1+i}{\sqrt{2}}, \quad z_1 = \frac{-1+i}{\sqrt{2}}, \quad z_2 = \frac{-1-i}{\sqrt{2}}, \quad z_3 = \frac{1-i}{\sqrt{2}}$$

Only z_0 and z_3 lie inside the circle $(x-1)^2+y^2=1$.

Now, we know that if sn snslytic function is of the form $f(z) = \dfrac{P(z)}{Q(z)}$, where

$P(z)$ and $Q(z)$ are rational functions then,

$$\text{Res }(f, z_k) = \lim_{z \to z_k} \frac{P(z)}{Q(z)} = \frac{P(z_k)}{Q'(z_k)}$$

Therefore,

$$\text{Res }(f, z_0) = \frac{1}{4(e^{i\pi/4})^3} = \frac{e^{-i3\pi/4}}{4} = \frac{1}{4}\left(-\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)$$

$$\text{Res }(f, z_3) = \frac{1}{4(e^{i7\pi/4})^3} = \frac{e^{-i21\pi/4}}{4} = \frac{e^{-i5\pi/4}}{4} = \frac{1}{4}\left(-\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)$$

By the Residue Theorem:

$$\int_C \frac{dz}{z^4+1} = 2\pi i\left(-\frac{2}{4\sqrt{2}}\right) = -\frac{\pi i}{\sqrt{2}}$$

(b) The contour is a circle centered at 2 with radius 0.5. The poles of the integrand $f(z) = \dfrac{z}{z^4+1}$ are the same as in part (a), all of which have a modulus $|z| = 1$. The distance from the center of the contour ($z = 2$) to any pole is at least $|2-1| = 1$. Since the radius of C is only 0.5, all poles lie outside the contour. By the Cauchy-Goursat Theorem, the integral of an analytic function over a closed contour is zero:

$$\int_C \frac{z\,dz}{z^4+1} = 0$$

(c) The poles are roots of $2z^4 + 1 = 0 \implies z^4 = -1/2$. The modulus of these poles is $|z| = (1/2)^{1/4} \approx 0.84$. Since $0.84 < 1$, all four poles lie inside the unit circle $|z| = 1$. Using the residue formula for $f(z) = \dfrac{P(z)}{Q(z)}$:

$$\text{Res }(f, z_k) = \frac{P(z_k)}{Q'(z_k)} = \frac{z_k^3}{8z_k^3} = \frac{1}{8}$$

Sum of residues $= 4 \times \dfrac{1}{8} = \dfrac{1}{2}$. By the Residue Theorem:

$$\int_C \frac{z^3\,dz}{2z^4+1} = 2\pi i\left(\frac{1}{2}\right) = \pi i$$

Q 65. Show that if $f(z)$ is analytic and $f(z) \neq 0$ in a simply connected domain Ω, then a single valued analytic branch of $f(z)^{\frac{1}{2}}$ can be defined in Ω.

Solution: To define the square root, we first establish a single-valued branch of the logarithm $\log f(z)$.

- Define $g(z) = \frac{f'(z)}{f(z)}$. Since f is analytic and non-vanishing in Ω, $g(z)$ is also analytic in Ω.
- Because Ω is simply connected, the line integral of $g(z)$ is independent of the path. Fix $z_0 \in \Omega$ and choose a constant c such that $e^c = f(z_0)$. We define:

$$h(z) = \int_{z_0}^{z} \frac{f'(\zeta)}{f(\zeta)} \, d\zeta + c$$

  By the Fundamental Theorem of Calculus for analytic functions, $h(z)$ is analytic in Ω and $h'(z) = \frac{f'(z)}{f(z)}$.

- Consider the function $\phi(z) = f(z)e^{-h(z)}$. Differentiating gives:

$$\phi'(z) = f'(z)e^{-h(z)} - f(z)h'(z)e^{-h(z)} = e^{-h(z)} \left(f'(z) - f(z)\frac{f'(z)}{f(z)} \right) = 0$$

  Thus, $\phi(z)$ is constant. At z_0, $\phi(z_0) = f(z_0)e^{-c} = 1$. Therefore, $e^{h(z)} = f(z)$, making $h(z)$ a single-valued analytic branch of $\log f(z)$.

- We define the function $G(z)$ as:

$$G(z) = e^{h(z)/2}$$

  Since $h(z)$ is analytic in Ω and the exponential function is entire, $G(z)$ is analytic in Ω.

- Squaring $G(z)$ yields:

$$[G(z)]^2 = \left(e^{h(z)/2}\right)^2 = e^{h(z)} = f(z)$$

  Thus, $G(z)$ is a single-valued analytic branch of $f(z)^{1/2}$ in Ω.

Q 66. Compute the residues at the singularities of $f(z) = \frac{\cos z}{z^2(z-\pi)^3}$.

Solution: To find the residues of the function

$$f(z) = \frac{\cos z}{z^2(z-\pi)^3}$$

we first identify the singularities and their orders. The function has:

- A **pole of order 2** at $z = 0$.

- A **pole of order 3** at $z = \pi$.

Residue at $z = 0$ (Pole of order 2)
Using the formula for a pole of order n:

$$\text{Res }(f, z_0) = \lim_{z \to z_0} \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} \left[(z - z_0)^n f(z) \right]$$

For $z = 0$ and $n = 2$:

$$\text{Res }(f, 0) = \lim_{z \to 0} \frac{d}{dz} \left[\frac{\cos z}{(z-\pi)^3} \right]$$

Applying the quotient rule:

$$\frac{d}{dz} \left[\frac{\cos z}{(z-\pi)^3} \right] = \frac{-(z-\pi)^3 \sin z - 3(z-\pi)^2 \cos z}{(z-\pi)^6} = \frac{-(z-\pi) \sin z - 3\cos z}{(z-\pi)^4}$$

Evaluating at $z = 0$:

$$\text{Res}(f, 0) = \frac{-(-\pi)(0) - 3(1)}{(-\pi)^4} = -\frac{3}{\pi^4}$$

Residue at $z = \pi$ (Pole of order 3)
For $z = \pi$ and $n = 3$:

$$\text{Res }(f, \pi) = \lim_{z \to \pi} \frac{1}{2!} \frac{d^2}{dz^2} \left[\frac{\cos z}{z^2} \right]$$

Let $g(z) = z^{-2} \cos z$. Computing the derivatives:

$$g'(z) = -2z^{-3} \cos z - z^{-2} \sin z$$

$$\begin{aligned} g''(z) &= 6z^{-4} \cos z + 2z^{-3} \sin z + 2z^{-3} \sin z - z^{-2} \cos z \\ &= \frac{6 \cos z}{z^4} + \frac{4 \sin z}{z^3} - \frac{\cos z}{z^2} \end{aligned}$$

Evaluating at $z = \pi$, noting that $\cos \pi = -1$ and $\sin \pi = 0$:

$$\text{Res }(f, \pi) = \frac{1}{2} \left[\frac{6(-1)}{\pi^4} + 0 - \frac{-1}{\pi^2} \right] = \frac{1}{2} \left[-\frac{6}{\pi^4} + \frac{1}{\pi^2} \right] = \frac{\pi^2 - 6}{2\pi^4}$$

Q 67. Evaluate each of the integral using Cauchy Residue Theorem.

(a) $\displaystyle\int_{|z|=5} \frac{\sin z}{z^2 - 4} \, dz$

(b) $\displaystyle\int_{|z|=3} \frac{e^{iz}}{z^2(z-2)(z+5i)} \, dz$

Solution: The Cauchy Residue Theorem states that if $f(z)$ is analytic inside and on a simple closed contour C, except for a finite number of isolated singularities $z_1, z_2, \ldots, z_n$ inside C, then:

$$\oint_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \text{Res}(f, z_k)$$

(a) The denominator $z^2 - 4 = 0$ yields roots at $z = 2$ and $z = -2$. The contour is a circle centered at the origin with radius $R = 5$. Since both poles lie **inside** the contour.
Using the formula for simple poles $\text{Res}(f, z_0) = \dfrac{P(z_0)}{Q'(z_0)}$:

$$\text{Res}\ (f, 2) = \left.\frac{\sin z}{2z}\right|_{z=2} = \frac{\sin(2)}{4}$$
$$\text{Res}\ (f, -2) = \left.\frac{\sin z}{2z}\right|_{z=-2} = \frac{\sin(-2)}{4} = \frac{-\sin(2)}{4}$$

Then,

$$\int_{|z|=5} \frac{\sin z}{z^2 - 4}\,dz = 2\pi i \left(\frac{\sin(2)}{4} - \frac{\sin(2)}{4}\right) = 0$$

(b) The integrand is given by $f(z) = \dfrac{e^{iz}}{z^2(z-2)(z+5i)}$. The singularities are the roots of the denominator:

- $z = 0$ (Pole of order 2)
- $z = 2$ (Simple pole)
- $z = -5i$ (Simple pole)

The contour is a circle centered at the origin with radius $R = 3$. We determine which poles lie inside the contour $|z| < 3$:

- $|0| = 0 < 3 \quad$ (Inside)
- $|2| = 2 < 3 \quad$ (Inside)
- $|-5i| = 5 > 3 \quad$ (Outside)

By the Cauchy Residue Theorem:

$$I = 2\pi i \left[\text{Res}(f, 0) + \text{Res}(f, 2)\right]$$

$$\text{Res}\ (f, 0) = \lim_{z\to 0} \frac{d}{dz}\left[z^2 f(z)\right] = \lim_{z\to 0} \frac{d}{dz}\left[\frac{e^{iz}}{(z-2)(z+5i)}\right]$$

$$
\begin{aligned}
&= \frac{ie^0(-10i) - e^0(2(0) - 2 + 5i)}{(-10i)^2} \\
&= \frac{10 - (-2 + 5i)}{-100} \\
&= \frac{12 - 5i}{-100} = -\frac{3}{25} + \frac{i}{20}
\end{aligned}
$$

$$\text{Res}(f, 2) = \lim_{z \to 2}(z-2)f(z) = \frac{e^{2i}}{2^2(2+5i)} = \frac{e^{2i}}{4(2+5i)} = \frac{e^{2i}}{58} - \frac{5ie^{2i}}{116}$$

Therefore,

$$\int_{|z|=3} \frac{e^{iz}}{z^2(z-2)(z+5i)}\,dz = 2\pi i\left(-\frac{3}{25} + \frac{i}{20} + \frac{e^{2i}(2-5i)}{116}\right)$$

Q 68. Using Rouche's Theorem find the number of roots of the given equations lying within the circle $|z| = 1$.

(a) $z^9 - 2z^6 + z^2 - 8z - 2 = 0$

(b) $z^7 - 5z^4 + z^2 - 2 = 0$

Solution:

(a) Consider the polynomial $P(z) = z^9 - 2z^6 + z^2 - 8z - 2$ on the boundary $C : |z| = 1$. We split $P(z)$ into two functions $f(z)$ and $g(z)$ on C such that:

- Let $f(z) = -8z$.
- Let $g(z) = z^9 - 2z^6 + z^2 - 2$.

On the unit circle $|z| = 1$:

- $|f(z)| = |-8z| = 8|z| = 8$.
- Using the **triangle inequality**:

$$|g(z)| = |z^9 - 2z^6 + z^2 - 2| \le |z|^9 + 2|z|^6 + |z|^2 + 2 = 1 + 2 + 1 + 2 = 6$$

Since $|f(z)| = 8 > 6 \ge |g(z)|$ for all $|z| = 1$, by **Rouché's Theorem**, $f(z)$ and $f(z) + g(z)$ have the same number of zeros inside $|z| < 1$. The function $f(z) = -8z$ has exactly **1 zero** (at $z = 0$).
There is exactly **1 root** inside the circle.

(b) Consider the polynomial $P(z) = z^7 - 5z^4 + z^2 - 2$ on the boundary $C : |z| = 1$. We split $P(z)$ as follows:

- Let $f(z) = -5z^4$.
- Let $g(z) = z^7 + z^2 - 2$.

On the unit circle $|z| = 1$:

- $|f(z)| = |-5z^4| = 5|z|^4 = 5$.
- Using the triangle inequality:

$$|g(z)| = |z^7 + z^2 - 2| \leq |z|^7 + |z|^2 + 2 = 1 + 1 + 2 = 4$$

Since $|f(z)| = 5 > 4 \geq |g(z)|$ for all $|z| = 1$, by Rouché's Theorem, $P(z)$ has the same number of zeros as $f(z)$ inside the circle. The function $f(z) = -5z^4$ has a **zero of multiplicity 4** at $z = 0$.
There are **4 roots** inside the circle

Q 69. How many roots does the equation $z = \phi(z)$ have in the circle $|z| < 1$, if for $|z| \leq 1$ the function $\phi(z)$ is analytic and satisfies the inequality $|\phi(z)| < 1$?

Solution: To determine the number of roots of the equation $z = \phi(z)$ (or $z - \phi(z) = 0$) in the circle $|z| < 1$, we use **Rouché's Theorem**. Let the two functions be:

- $f(z) = z$
- $h(z) = -\phi(z)$

We consider the sum $g(z) = f(z) + h(z) = z - \phi(z)$. We want to find the number of zeros of $g(z)$ in the region $|z| < 1$.
Rouché's Theorem states that if $f(z)$ and $h(z)$ are analytic inside and on a simple closed contour C, and if $|h(z)| < |f(z)|$ on C, then $f(z)$ and $f(z) + h(z)$ have the same number of zeros inside C.
On the boundary $|z| = 1$:

- The modulus of the first function is $|f(z)| = |z| = 1$.
- We are given that for $|z| \leq 1$, $|\phi(z)| < 1$. Thus, $|h(z)| = |-\phi(z)| < 1$ on the boundary.

Therefore, on the boundary $|z| = 1$, we have, $|h(z)| < |f(z)|$. Since the inequality holds on the boundary, $g(z) = z - \phi(z)$ has the same number of zeros inside $|z| < 1$ as $f(z) = z$.

The function $f(z) = z$ has exactly **one zero** (at $z = 0$) inside the unit circle. Thus, the equation $z = \phi(z)$ has exactly **one root** in the circle $|z| < 1$.

Q 70. If $\lim_{n\to\infty} z_n = A$, prove that $\lim_{n\to\infty} \frac{1}{n}(z_1 + z_2 + \cdots + z_n) = A$.

Solution: Let $\epsilon > 0$ be given. We wish to show that there exists an integer M such that for all $n > M$:

$$\left| \frac{1}{n} \sum_{k=1}^{n} z_k - A \right| < \epsilon$$

Observe that we can write $A = \frac{1}{n} \sum_{k=1}^{n} A$. Therefore:

$$\left| \frac{1}{n} \sum_{k=1}^{n} z_k - A \right| = \left| \frac{1}{n} \sum_{k=1}^{n} (z_k - A) \right| \leq \frac{1}{n} \sum_{k=1}^{n} |z_k - A|$$

Since $\lim_{n\to\infty} z_n = A$, there exists a positive integer N such that for all $k > N$:

$$|z_k - A| < \frac{\epsilon}{2}$$

For $n > N$, we split the sum into a constant head and a small tail:

$$\frac{1}{n} \sum_{k=1}^{n} |z_k - A| = \frac{1}{n} \sum_{k=1}^{N} |z_k - A| + \frac{1}{n} \sum_{k=N+1}^{n} |z_k - A|$$

- **The Tail:** Since $|z_k - A| < \frac{\epsilon}{2}$ for all $k > N$:

$$\frac{1}{n} \sum_{k=N+1}^{n} |z_k - A| < \frac{1}{n} \sum_{k=N+1}^{n} \frac{\epsilon}{2} = \frac{n - N}{n} \cdot \frac{\epsilon}{2} < \frac{\epsilon}{2}$$

- **The Head:** Let $C = \sum_{k=1}^{N} |z_k - A|$ be a fixed constant. Since $\lim_{n\to\infty} \frac{C}{n} = 0$, there exists N_1 such that for all $n > N_1$:

$$\frac{1}{n} \sum_{k=1}^{N} |z_k - A| < \frac{\epsilon}{2}$$

Let $M = \max(N, N_1)$. For all $n > M$, we have:

$$\left| \frac{1}{n} \sum_{k=1}^{n} z_k - A \right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Thus, $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} z_k = A$.

Q 71. Expand $(1-z)^{-m}$, where m is a positive integer, in powers of z.

Solution: To expand $(1-z)^{-m}$ where m is a positive integer, we use the generalized binomial theorem. The expansion is valid for $|z| < 1$. The General Formula Binomial theorem says that,

$$(1+x)^n = \sum_{k=0}^{\infty} \binom{n}{k} x^k$$

Using the binomial theorem for a negative exponent $n = -m$ and substituting $-z$ for x:

$$(1-z)^{-m} = \sum_{k=0}^{\infty} \binom{-m}{k} (-z)^k$$

Applying the property of upper negation, $\binom{-m}{k} = (-1)^k \binom{m+k-1}{k}$, we get:

$$(1-z)^{-m} = \sum_{k=0}^{\infty} (-1)^k \binom{m+k-1}{k} (-1)^k z^k = \sum_{k=0}^{\infty} \binom{m+k-1}{k} z^k$$

Q 72. Expand $\frac{2z+3}{z+1}$ in powers of $z-1$. What is the radius of convergence?

Solution: To expand the function $f(z) = \frac{2z+3}{z+1}$ around the point $a = 1$, we substitute $u = z-1$, which implies $z = u+1$ and rewrite the function to isolate the constant and simplify the fraction:

$$f(z) = \frac{2z+1}{z+1} = \frac{2(u+1)+3}{u+1+1} = \frac{2u+5}{u+2} = 2 + \frac{1}{u+2}$$

We know by the geometric series formula $\frac{1}{1-r} = \sum_{n=0}^{\infty} r^n$ for $|r| < 1$. Then,

$$\frac{1}{u+2} = \frac{1}{2(1-(-\frac{u}{2}))} = \frac{1}{2} \sum_{n=0}^{\infty} \left(-\frac{u}{2}\right)^n \quad , \text{for} \quad \left|-\frac{u}{2}\right| < 1$$

Now substituting this back into the expression for $f(z)$ we get,

$$f(z) = 2 + \frac{1}{2} \sum_{n=0}^{\infty} \left(-\frac{z-1}{2}\right)^n = 2 + \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} (z-1)^n$$

where, $|z-1|<2$. Thus the series converges when $|z-1|<2$. The radius of convergence is $R=2$. This matches the distance from the center $z=1$ to the singularity at $z=-1$.

Q 73. Find the radius of convergence:

(a) $\sum z^{n!}$

(b) $\sum n^p z^n$

(c) $\sum \frac{z^n}{n!}$

(d) $\sum n! z^n$

(e) $\sum q^{n^2} z^n, \quad (|q|<1)$

Solution: The radius of convergence R for a power series $\sum a_n z^n$ is determined using the **Cauchy-Hadamard Theorem**:

$$\frac{1}{R}=\limsup_{n\to\infty}|a_n|^{1/n} \quad \text{or by the Ratio Test: } R=\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$

(a) Here, the coefficients $a_k=1$ if $k=n!$ and $a_k=0$ otherwise.

$$\frac{1}{R}=\limsup_{k\to\infty}|a_k|^{1/k}=\lim_{n\to\infty}(1)^{1/n!}=1 \implies \mathbf{R=1}$$

(b) Using the Ratio Test:

$$R=\lim_{n\to\infty}\left|\frac{n^p}{(n+1)^p}\right|=\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^p=1^p=1 \implies \mathbf{R=1}$$

(c) Using the Ratio Test:

$$R=\lim_{n\to\infty}\left|\frac{1/n!}{1/(n+1)!}\right|=\lim_{n\to\infty}\frac{(n+1)!}{n!}=\lim_{n\to\infty}(n+1)=\infty \implies \mathbf{R=\infty}$$

(d) Using the Ratio Test:

$$R=\lim_{n\to\infty}\left|\frac{n!}{(n+1)!}\right|=\lim_{n\to\infty}\frac{1}{n+1}=0 \implies \mathbf{R=0}$$

(e) Using the Root Test:

$$\frac{1}{R} = \limsup_{n\to\infty} |q^{n^2}|^{1/n} = \limsup_{n\to\infty} |q|^n$$

Since $|q| < 1$, $\lim_{n\to\infty} |q|^n = 0$. Thus, $1/R = 0 \implies \mathbf{R} = \infty$.

Q 74. If $\sum a_n z^n$ has radius of convergence R, what is the radius of convergence of $\sum a_n z^{2n}$ and $\sum a_n^2 z^n$?

Solution: Given that the power series $\sum a_n z^n$ has a radius of convergence R, we apply the **Cauchy-Hadamard Theorem**, which states:

$$\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{1/n}$$

(a) Let $w = z^2$. The series becomes $\sum a_n w^n$.

- This transformed series $\sum a_n w^n$ converges when $|w| < R$.
- Substituting back for z, the condition becomes $|z^2| < R$.
- Taking the square root of both sides, we find $|z| < \sqrt{R}$.

Thus, the radius of convergence for $\sum a_n z^{2n}$ is $\sqrt{\mathbf{R}}$.

(b) Let R' be the radius of convergence for the series $\sum a_n^2 z^n$. Applying the Cauchy-Hadamard formula:

$$\frac{1}{R'} = \limsup_{n\to\infty} |a_n^2|^{1/n} = \limsup_{n\to\infty} \left(|a_n|^{1/n}\right)^2$$

Since $\limsup_{n\to\infty} |a_n|^{1/n} = \frac{1}{R}$, we substitute:

$$\frac{1}{R'} = \left(\frac{1}{R}\right)^2 = \frac{1}{R^2} \implies R' = R^2$$

Thus, the radius of convergence for $\sum a_n^2 z^n$ is $\mathbf{R^2}$.

Q 75. If $f(z) = \sum a_n z^n$, what is $\sum n^3 a_n z^n$?

Solution: Let,

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

$$\begin{aligned}
\implies \quad & f'(z) &&= \sum_{n=1}^{\infty} n a_n z^{n-1} \\
\implies \quad & z f'(z) &&= \sum_{n=0}^{\infty} n a_n z^{n}. \\
\implies \quad & z f''(z) + f'(z) &&= \sum_{n=1}^{\infty} n^2 a_n z^{n-1} \\
\implies \quad & z^2 f''(z) + z f'(z) &&= \sum_{n=0}^{\infty} n^2 a_n z^{n}. \\
\implies \quad & z^2 f'''(z) + 2z f''(z) + z f''(z) + f'(z) &&= \sum_{n=0}^{\infty} n^3 a_n z^{n-1}. \\
\implies \quad & z^3 f'''(z) + 3z^2 f''(z) + z f'(z) &&= \sum_{n=0}^{\infty} n^3 a_n z^{n}.
\end{aligned}$$

Therefore,

$$\sum_{n=0}^{\infty} n^3 a_n z^n = z f'(z) + 3z^2 f''(z) + z^3 f'''(z)$$

Q 76. If $\lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|} = R$, prove that $\sum a_n z^n$ has radius of convergence R.

Solution: Given the power series $\sum_{n=0}^{\infty} a_n z^n$ and the limit:

$$\lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|} = R$$

We want to prove that the radius of convergence of this series is R.
A power series $\sum a_n z^n$ converges absolutely if the limit of the ratio of successive terms is less than 1. Let L be this limit:

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1} z^{n+1}}{a_n z^n} \right|$$

Factoring out the term $|z|$, which is independent of n, we obtain:

$$L = |z| \cdot \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

From the given condition, we know:

$$\lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|} = R \implies \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{1}{R}$$

Substituting the limit back into our expression for L:

$$L = |z| \cdot \frac{1}{R}$$

According to the Ratio Test, the series converges when $L < 1$:

$$\frac{|z|}{R} < 1 \implies |z| < R$$

Similarly, the series diverges when $L > 1$, which occurs when $|z| > R$.
By the definition of the radius of convergence, the series $\sum a_n z^n$ converges for all $|z|$ within a disk of radius R and diverges outside of it. Thus, the radius of convergence is R.

Q 77. For what value of z is the series $\displaystyle\sum_{n=0}^{\infty}\left(\frac{z}{1+z}\right)^n$ convergent?

Solution: To determine the values of z for which the series $\displaystyle\sum_{n=0}^{\infty}\left(\frac{z}{1+z}\right)^n$ converges, we treat it as a **geometric series**.
A geometric series of the form $\displaystyle\sum_{n=0}^{\infty} r^n$ converges if and only if the absolute value of the common ratio r is strictly less than 1. In this case:

$$r = \frac{z}{1+z}$$

Therefore, the condition for convergence is:

$$\left|\frac{z}{1+z}\right| < 1 \implies |z| < |1+z|$$

Then, letting $z = x + iy$

$$\begin{aligned}
|z|^2 &< |1+z|^2 \\
\implies x^2 + y^2 &< (x+1)^2 + y^2 \\
\implies x^2 + y^2 &< x^2 + 2x + 1 + y^2 \\
\implies 0 &< 2x + 1 \\
\implies 2x &> -1 \\
\implies x &> -\frac{1}{2}
\end{aligned}$$

The series converges for all complex values z such that $\text{Re}\,(z) > -\dfrac{1}{2}$

Q 78. Let $C = \{z : |z| = 1\}$ be the unit circle, oriented counter clockwise. For any $z \in \mathbb{C}$, with $|z| \neq 1$, evaluate $\displaystyle\int_C \frac{\overline{\zeta}}{\zeta - z}\, d\zeta$.

Solution: Let $C = \{z : |z| = 1\}$ be the unit circle, oriented counter-clockwise. For $|z| \neq 1$, we evaluate:

$$I = \int_C \frac{\overline{\zeta}}{\zeta - z}\, d\zeta$$

On the unit circle C, the variable ζ satisfies $|\zeta| = 1$, which implies $\zeta\overline{\zeta} = 1$. Therefore, we can substitute $\overline{\zeta} = \dfrac{1}{\zeta}$:

$$I = \int_C \frac{1/\zeta}{\zeta - z}\, d\zeta = \int_C \frac{1}{\zeta(\zeta - z)}\, d\zeta$$

Let $f(\zeta) = \dfrac{1}{\zeta(\zeta - z)}$. The function has two simple poles at $\zeta = 0$ and $\zeta = z$. The residues are calculated as follows:

- **Residue at $\zeta = 0$:** $\text{Res}\,(f, 0) = \lim_{\zeta \to 0} \zeta \cdot \dfrac{1}{\zeta(\zeta - z)} = -\dfrac{1}{z}$
- **Residue at $\zeta = z$:** $\text{Res}\,(f, z) = \lim_{\zeta \to z} (\zeta - z) \cdot \dfrac{1}{\zeta(\zeta - z)} = \dfrac{1}{z}$

By the Residue Theorem, the integral is $I = 2\pi i \times$ (sum of residues enclosed by C).

- **Case 1:** $|z| < 1$**:** Both poles $\zeta = 0$ and $\zeta = z$ are inside the unit circle C.

$$I = 2\pi i\,(\text{Res}\,(f, 0) + \text{Res}\,(f, z)) = 2\pi i \left(-\frac{1}{z} + \frac{1}{z}\right) = 0$$

- **Case 2:** $|z| > 1$**:** Only the pole $\zeta = 0$ is inside the unit circle C. The pole $\zeta = z$ lies outside.

$$I = 2\pi i\,(\text{Res}(f, 0)) = 2\pi i \left(-\frac{1}{z}\right) = -\frac{2\pi i}{z}$$

Therefore,

$$\int_C \frac{\overline{\zeta}}{\zeta - z}\, d\zeta = \begin{cases} 0 & \text{if } |z| < 1 \\ -\dfrac{2\pi i}{z} & \text{if } |z| > 1 \end{cases}$$

Q 79. Let $\mathbb{D} = \{z : |z| = 1\}$ and let $f : \mathbb{D} \to \mathbb{D}$ be analytic. Suppose that there are two points $a, b \in \mathbb{D}$ with $a \neq b$, such that $f(a) = a$ and $f(b) = b$. Prove that $f(z) = z$ for all $z \in \mathbb{D}$.

Solution: To prove this, we will use a Möbius transformation to move one fixed point to the origin and then apply the **Schwarz Lemma**.
Let $\psi : \mathbb{D} \to \mathbb{D}$ be an automorphism of the unit disk (a Möbius transformation) defined by:

$$\psi(z) = \frac{z - a}{1 - \overline{a}z}$$

This map is a bijection from $\mathbb{D}$ to $\mathbb{D}$ such that $\psi(a) = 0$. Let ψ^{-1} be its inverse.
Define the function $g : \mathbb{D} \to \mathbb{D}$ by:

$$g = \psi \circ f \circ \psi^{-1}$$

Since ψ, f, and ψ^{-1} are analytic self-maps of the unit disk, g is also analytic on $\mathbb{D}$.
We examine the behavior of g at two specific points:

- At the origin: $g(0) = \psi(f(\psi^{-1}(0))) = \psi(f(a)) = \psi(a) = 0$.
- At $c = \psi(b)$: Since $a \neq b$ and ψ is injective, $c \neq 0$. Furthermore:

$$g(c) = \psi(f(\psi^{-1}(c))) = \psi(f(b)) = \psi(b) = c$$

Thus, g is an analytic self-map of the disk that fixes the origin and a non-zero point c.
The Schwarz Lemma states that if $g : \mathbb{D} \to \mathbb{D}$ is analytic with $g(0) = 0$, then $|g(z)| \leq |z|$ for all $z \in \mathbb{D}$. Moreover, if $|g(z_0)| = |z_0|$ for any $z_0 \neq 0$, then $g(z) = e^{i\theta}z$ for some constant $\theta \in \mathbb{R}$.
In our case, $g(c) = c$ implies $|g(c)| = |c|$. Since $c \neq 0$, g must be a rotation:

$$g(z) = e^{i\theta}z$$

Substituting the fixed point c back into this equation:

$$c = e^{i\theta}c \implies e^{i\theta} = 1$$

This implies $g(z) = z$ for all $z \in \mathbb{D}$.
Since g is the identity map, we have $\psi \circ f \circ \psi^{-1} = \text{id}$. It follows that:

$$f = \psi^{-1} \circ \text{id} \circ \psi = \text{id}$$

Thus, $f(z) = z$ for all $z \in \mathbb{D}$, which completes the proof.

Q 80. Expand the given functions in a Laurent series in a neighborhood of the given points. Determine the domain within which the expansion holds.

(a) $\dfrac{1}{z(1-z)}$ in the neighborhood of $z = 0$ and $z = 1$

(b) $\dfrac{1}{(z^2+1)^2}$ in the neighborhood of $z = i$

(c) $z^2 e^{1/z}$ in the neighborhood of $z = 0$

Solution:

(a) $f(z) = \dfrac{1}{z(1-z)}$

Neighborhood of $z = 0$: We use the geometric series expansion for $\dfrac{1}{1-z}$, which is valid for $|z| < 1$:

$$f(z) = \frac{1}{z} \cdot \frac{1}{1-z} = \frac{1}{z} \sum_{n=0}^{\infty} z^n$$

Domain: $0 < |z| < 1$.

Neighborhood of $z = 1$: Let $u = z - 1$, so $z = u + 1$. The function becomes:

$$f(z) = \frac{1}{(u+1)(-u)} = -\frac{1}{u} \cdot \frac{1}{1+u}$$

Expanding $\dfrac{1}{1+u}$ as a geometric series $\sum_{n=0}^{\infty}(-u)^n$ for $|u| < 1$:

$$f(z) = -\frac{1}{u} \sum_{n=0}^{\infty} (-1)^n u^n = \sum_{n=0}^{\infty} (-1)^{n+1} u^{n-1}$$

Substituting back $u = z - 1$:

$$f(z) = \sum_{n=0}^{\infty} (-1)^{n+1} (z-1)^{n-1}$$

Domain: $0 < |z - 1| < 1$.

(b) $f(z) = \dfrac{1}{(z^2+1)^2}$ in the neighborhood of $z = i$

Factor the denominator: $f(z) = \dfrac{1}{(z-i)^2(z+i)^2}$. Let $u = z - i$, then $z + i = u + 2i$:

$$f(z) = \frac{1}{u^2(u+2i)^2} = \frac{1}{u^2(2i)^2\left(1 + \frac{u}{2i}\right)^2} = \frac{1}{u^2(2i)^2}\left(1 - \left(-\frac{u}{2i}\right)\right)^{-2}$$

Using the binomial expansion $(1-x)^{-2} = \sum_{n=0}^{\infty}(n+1)x^n$:

$$f(z) = \frac{1}{u^2(2i)^2}\sum_{n=0}^{\infty}(n+1)\left(-\frac{u}{2i}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n(n+1)}{(2i)^{n+2}}(z-i)^{n-2}$$

Domain: The nearest singularity is at $z=-i$. The distance $|i-(-i)| = 2$. Thus, $0 < |z-i| < 2$.

(c) $f(z) = z^2 e^{1/z}$ in the neighborhood of $z = 0$

We use the Taylor series for $e^w = \sum_{n=0}^{\infty}\frac{w^n}{n!}$ with $w = \frac{1}{z}$:

$$e^{1/z} = \sum_{n=0}^{\infty}\frac{1}{n!z^n}$$

Then,

$$f(z) = z^2e^{1/z} = z^2\sum_{n=0}^{\infty}\frac{1}{n!z^n} = \sum_{n=0}^{\infty}\frac{1}{n!z^{n-2}}$$

Domain: The only singularity is at $z = 0$, so the expansion is valid for $|z| > 0$.

Q 81. Expand the given functions in a Laurent series in the given ring.

(a) $e^{2z} + \cos\frac{2}{z}$ in the domain $\mathbb{C}\setminus\{0\}$

(b) $\dfrac{1}{(z-1)(z-2)}$ in the ring $1 < |z| < 2$

Solution:

(a) **Function:** $f(z) = e^{2z} + \cos\dfrac{2}{z}$ in the domain $\mathbb{C}\setminus\{0\}$.

Using the standard Maclaurin series for the exponential and cosine functions:

- $e^{2z} = \sum_{n=0}^{\infty}\dfrac{(2z)^n}{n!} = 1 + \sum_{n=1}^{\infty}\dfrac{2^n}{n!}z^n$
- $\cos\dfrac{2}{z} = \sum_{n=0}^{\infty}\dfrac{(-1)^n(2/z)^{2n}}{(2n)!} = 1 + \sum_{n=1}^{\infty}\dfrac{(-1)^n 2^{2n}}{(2n)!}z^{-2n}$

Summing these together, the constant terms $1 + 1 = 2$ combine to give the Laurent series:

$$f(z) = 2 + \sum_{n=1}^{\infty}\frac{2^n}{n!}z^n + \sum_{n=1}^{\infty}\frac{(-1)^n2^{2n}}{(2n)!}z^{-2n}$$

Continued...

(b) **Function:** $f(z) = \dfrac{1}{(z-1)(z-2)}$ in the ring $1 < |z| < 2$.
We can write the function as:

$$f(z) = \frac{1}{z-2} - \frac{1}{z-1}$$

Now expand each term based on the domain $1 < |z| < 2$:

- For $\dfrac{1}{z-2}$, since $|z| < 2$, we have $|\dfrac{z}{2}| < 1$:

$$\frac{1}{z-2} = -\frac{1}{2} \cdot \frac{1}{1-\frac{z}{2}} = -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n = -\sum_{n=0}^{\infty}\frac{1}{2^{n+1}}z^n$$

- For $-\dfrac{1}{z-1}$, since $|z| > 1$, we have $|\dfrac{1}{z}| < 1$:

$$-\frac{1}{z-1} = -\frac{1}{z} \cdot \frac{1}{1-\frac{1}{z}} = -\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n = -\sum_{n=0}^{\infty} z^{-(n+1)} = -\sum_{n=1}^{\infty} z^{-n}$$

Therefore,

$$f(z) = -\sum_{n=1}^{\infty} z^{-n} - \sum_{n=0}^{\infty}\frac{1}{2^{n+1}}z^n$$

Q 82. Evaluate

(a) $\displaystyle\int_0^{2\pi} e^{\sin(e^{i\theta})}\, d\theta$

(b) $\displaystyle\int_0^{\pi} \frac{8d\theta}{5+2\cos\theta}$

Solution:

(a) Let $z = e^{i\theta}$. Then $dz = ie^{i\theta}d\theta = iz\,d\theta$, which gives $d\theta = \dfrac{dz}{iz}$. As θ ranges from 0 to 2π, z traverses the unit circle C in the counterclockwise direction.
Substituting these into the integral:

$$I = \oint_C e^{\sin(z)}\frac{dz}{iz} = \frac{1}{i}\oint_C \frac{e^{\sin(z)}}{z}\, dz$$

By Cauchy's Integral Formula, for a function $f(z)$ analytic inside and on C:

$$\oint_C \frac{f(z)}{z - z_0}\, dz = 2\pi i f(z_0)$$

Let $f(z) = e^{\sin(z)}$, which is an entire function. Taking $z_0 = 0$ (which lies inside the unit circle):

$$I = \frac{1}{i}\left[2\pi i \cdot e^{\sin(0)}\right] = 2\pi \cdot e^0 = 2\pi$$

(b) Since the integrand is symmetric about $\theta = \pi$ (because $\cos(2\pi - \theta) = \cos\theta$), we can relate it to the integral over the full circle:

$$I = \frac{1}{2}\int_0^{2\pi} \frac{8\, d\theta}{5 + 2\cos\theta} = \int_0^{2\pi} \frac{4\, d\theta}{5 + 2\cos\theta}$$

Using the complex substitution $z = e^{i\theta}$, we have $\cos\theta = \frac{1}{2}(z + z^{-1})$ and $d\theta = \frac{dz}{iz}$:

$$I = \oint_C \frac{4}{5 + 2\left(\frac{z + z^{-1}}{2}\right)} \frac{dz}{iz} = \frac{4}{i}\oint_C \frac{1}{z^2 + 5z + 1}\, dz$$

The roots of $z^2 + 5z + 1 = 0$ are $z = \frac{-5 \pm \sqrt{21}}{2}$. The root inside the unit circle is $z_1 = \frac{-5 + \sqrt{21}}{2}$. Using the Residue Theorem:

$$\text{Res}\,(f, z_1) = \left.\frac{1}{\frac{d}{dz}(z^2 + 5z + 1)}\right|_{z = z_1} = \frac{1}{2z_1 + 5} = \frac{1}{\sqrt{21}}$$

Thus:

$$I = \frac{4}{i}\left(2\pi i \cdot \frac{1}{\sqrt{21}}\right) = \frac{8\pi}{\sqrt{21}}$$

Q 83. Find all entire functions f such that f has a zero of order 2 at the origin, $f(i) = -2$ and $|f'(z)| \leq 6|z|$ or explain why such a function f cannot exist.

Solution: Let $g(z) = f'(z)$. Since $f(z)$ is an entire function, $g(z)$ is also an entire function. The condition f having a zero of order 2 at $z = 0$ implies $f(0) = 0$ and $f'(0) = 0$. Thus, $g(0) = 0$.
Consider the function $h(z)$ defined by:

$$h(z) = \begin{cases} \frac{f'(z)}{z} & z \neq 0 \\ f''(0) & z = 0 \end{cases}$$

Since $f'(z)$ is entire and $f'(0) = 0$, $h(z)$ has a removable singularity at $z = 0$ and is therefore an entire function.
From the given inequality $|f'(z)| \leq 6|z|$, we have:

$$|h(z)| = \frac{|f'(z)|}{|z|} \leq 6 \quad \text{for all } z \neq 0.$$

By **Liouville's Theorem**, any bounded entire function must be constant. Therefore, $h(z) = c$ for some constant $c \in \mathbb{C}$, where $|c| \leq 6$.
Substituting back for $f'(z)$:

$$\frac{f'(z)}{z} = c \implies f'(z) = cz$$

Integrating $f'(z)$ with respect to z:

$$f(z) = \int cz\, dz = \frac{c}{2}z^2 + d$$

Since $f(z)$ has a zero at $z = 0$, we have $f(0) = 0$, which implies $d = 0$. Thus:

$$f(z) = \frac{c}{2}z^2$$

Using the condition $f(i) = -2$:

$$f(i) = \frac{c}{2}(i)^2 = \frac{c}{2}(-1) = -\frac{c}{2}$$

Setting this equal to -2:

$$-\frac{c}{2} = -2 \implies c = 4$$

We check the bound $|c| \leq 6$: since $|4| \leq 6$, this value is valid.
Substituting $c = 4$ into the expression for $f(z)$:

$$f(z) = \frac{4}{2}z^2 = 2z^2$$

Therefore, the only such function is: $\mathbf{f(z) = 2z^2}$.

Q 84. Let $P(z) = 2z^4 + 5z^2$ and $Q(z) = z^4 + 10z^2 + 1$. Prove that P and Q have the same number of zeroes inside the open disk as well as the same number of zeroes outside the unit disk but inside the disk of radius 4 centered at 0.

Solution: Let $P(z) = 2z^4 + 5z^2$ and $Q(z) = z^4 + 10z^2 + 1$. We wish to show they have the same number of zeros in the regions $|z| < 1$ and $1 < |z| < 4$.

(a) Zeros inside the unit disk ($|z| < 1$)
For $Q(z)$, let $f(z) = 10z^2$ and $g(z) = z^4 + 1$. On the boundary $|z| = 1$:

$$|f(z)| = 10|z|^2 = 10$$

$$|g(z)| = |z^4 + 1| \leq |z|^4 + 1 = 2$$

Since $|g(z)| < |f(z)|$ on $|z| = 1$, Rouché's Theorem implies $Q(z) = f(z) + g(z)$ has the same number of zeros as $f(z) = 10z^2$ inside the disk, which is **2 zeros**. For $P(z) = z^2(2z^2 + 5)$, the zeros are $z = 0$ (multiplicity 2) and $z = \pm i\sqrt{2.5}$. Since $\sqrt{2.5} > 1$, only the origin is inside the disk. Thus, $P(z)$ has **2 zeros** inside $|z| < 1$.

(b) Zeros inside the disk of radius 4 ($|z| < 4$)
For $P(z)$, let $f(z) = 2z^4$ and $g(z) = 5z^2$. On the boundary $|z| = 4$:

$$|f(z)| = 2(4^4) = 512, \quad |g(z)| = 5(4^2) = 80$$

Since $80 < 512$, $P(z)$ has the same number of zeros as $2z^4$, which is **4 zeros**. For $Q(z)$, let $f(z) = z^4$ and $g(z) = 10z^2 + 1$. On the boundary $|z| = 4$:

$$|f(z)| = 4^4 = 256, \quad |g(z)| \leq 10(4^2) + 1 = 161$$

Since $161 < 256$, $Q(z)$ has the same number of zeros as z^4, which is **4 zeros**.

The number of zeros in the annulus $1 < |z| < 4$ is the number of zeros in the disk $|z| < 4$ minus those in the disk $|z| \leq 1$.

- For $P(z)$: $4 - 2 = \mathbf{2}$ zeros.
- For $Q(z)$: $4 - 2 = \mathbf{2}$ zeros.

Both functions have 2 zeros in $|z| < 1$ and 2 zeros in $1 < |z| < 4$.

Q 85. Let $\sum_{n=0}^{\infty} a_n z^n$ converge for the nonzero complex number $z = z_0$. Prove that $\sum_{n=0}^{\infty} a_n z^n$ converges to an analytic function for all z, such that $|z| < |z_0|$.

Solution: The proof is divided into two main parts: establishing absolute convergence and then applying the property of analyticity for power series.

- **Absolute Convergence:**
  Since the series $\sum_{n=0}^{\infty} a_n z_0^n$ converges, the sequence of its terms must approach

zero as $n \to \infty$. Consequently, the sequence $\{a_n z_0^n\}$ is bounded. There exists a real constant $M > 0$ such that:

$$|a_n z_0^n| \leq M \quad \text{for all } n \geq 0$$

Now, consider any z such that $|z| < |z_0|$. We can express the magnitude of the general term as:

$$|a_n z^n| = |a_n z_0^n| \left| \frac{z}{z_0} \right|^n \leq M \left| \frac{z}{z_0} \right|^n$$

Let $r = \left| \frac{z}{z_0} \right|$. Since $|z| < |z_0|$, it follows that $0 \leq r < 1$. The series $\sum_{n=0}^{\infty} M r^n$ is a geometric series that converges because $r < 1$. By the **Comparison Test**, the series $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely for all $|z| < |z_0|$.

- **Analyticity**
  The absolute convergence established above implies that the radius of convergence R of the power series satisfies $R \geq |z_0|$.
  Inside the disk of convergence $D = \{z : |z| < R\}$, a power series defines a function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ that is:
  - **Uniformly convergent** on any compact subset of D.
  - **Complex-differentiable (Holomorphic)** at every point in D, because the derivative can be calculated by term-by-term differentiation: $f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$.

  Since a function that is holomorphic on an open set is by definition **analytic**, the function $f(z)$ is analytic for all z such that $|z| < |z_0|$.

Q 86. Show that if z and z' correspond to diametrically opposite points on the Riemann sphere then $z\overline{z'} = -1$.

Solution: Consider the unit sphere $S^2 \subset \mathbb{R}^3$ defined by the equation $x^2 + y^2 + \zeta^2 = 1$. The stereographic projection from the North Pole $N = (0, 0, 1)$ maps a point $P = (x, y, \zeta)$ on the sphere to a complex number $z = u + iv$ in the equatorial plane $\zeta = 0$. The formulas for this projection are:

$$u = \frac{x}{1 - \zeta}, \quad v = \frac{y}{1 - \zeta}$$

Thus, the complex number z is given by:

$$z = \frac{x + iy}{1 - \zeta}$$

The diametrically opposite point to $P = (x, y, \zeta)$ is obtained by reflection through the origin:

$$P' = (-x, -y, -\zeta)$$

Applying the projection formula to P', we find the complex number z':

$$z' = \frac{-x - iy}{1 - (-\zeta)} = \frac{-(x + iy)}{1 + \zeta}$$

We now calculate the product $z\overline{z'}$. First, note that since x, y, ζ are real, the conjugate of z' is:

$$\overline{z'} = \frac{-(x - iy)}{1 + \zeta}$$

Multiplying z and $\overline{z}'$:

$$z\overline{z'} = \left(\frac{x + iy}{1 - \zeta}\right)\left(\frac{-(x - iy)}{1 + \zeta}\right)$$

$$z\overline{z'} = \frac{-(x^2 + y^2)}{(1 - \zeta)(1 + \zeta)} = \frac{-(x^2 + y^2)}{1 - \zeta^2}$$

From the equation of the sphere, we know $x^2 + y^2 + \zeta^2 = 1$, which implies $x^2 + y^2 = 1 - \zeta^2$. Substituting this into the equation above:

$$z\overline{z'} = \frac{-(1 - \zeta^2)}{1 - \zeta^2} = -1$$

Therefore, if z and z' are diametrically opposite points on the Riemann sphere, they satisfy the relation:

$$z\overline{z'} = -1 \quad \text{or} \quad z' = -\frac{1}{\overline{z}}$$

Q 87. Prove that there is no function f which is analytic on $\mathbb{D}$ such that $\lim\limits_{|z|\to 1} |f(z)| = \infty$.

Solution: Suppose such a function f exists. Since $\lim\limits_{|z|\to 1} |f(z)| = \infty$, there exists some $r \in (0, 1)$ such that $f(z) \neq 0$ for all z in the annulus $r < |z| < 1$. This implies that f can only have a finite number of zeros in $\mathbb{D}$.
Define a function $g : \mathbb{D} \to \mathbb{C}$ by:

$$g(z) = \frac{1}{f(z)}$$

The function g is analytic on $\mathbb{D}$ except at the finite number of zeros of f, where g has poles. However, the condition $\lim_{|z|\to 1} |f(z)| = \infty$ implies that:

$$\lim_{|z|\to 1} |g(z)| = \lim_{|z|\to 1} \frac{1}{|f(z)|} = 0$$

Since $g(z)$ is bounded as $|z| \to 1$, all its poles in $\mathbb{D}$ are actually removable singularities. By defining $g(z)$ at these points using their limits, we obtain a function $\tilde{g}$ that is analytic on the entire disk $\mathbb{D}$.
Because $\tilde{g}$ is analytic on $\mathbb{D}$ and continuous on the closure $\overline{\mathbb{D}}$ (with $\tilde{g}(z) = 0$ for $|z| = 1$), we can apply the **Maximum Modulus Principle**. The principle states that the maximum of $|\tilde{g}(z)|$ on $\overline{\mathbb{D}}$ must occur on the boundary.
Since $|\tilde{g}(z)| = 0$ for all z such that $|z| = 1$, it follows that:

$$|\tilde{g}(z)| \leq 0 \quad \text{for all } z \in \mathbb{D}$$

This implies $\tilde{g}(z) = 0$ for all $z \in \mathbb{D}$. But $\tilde{g}(z) = \dfrac{1}{f(z)}$, and a reciprocal of a complex number cannot be zero ($\dfrac{1}{f(z)} \neq 0$). This is a contradiction.
Therefore, no such analytic function f exists.

Q 88. Let f and g be entire functions that satisfy $f(0) = g(0) \neq 0$ and $|f(z)| \leq |g(z)|$, $\forall z \in \mathbb{C}$. Prove that $f = g$.

Solution: Let f and g be entire functions such that $f(0) = g(0) \neq 0$ and $|f(z)| \leq |g(z)|$ for all $z \in \mathbb{C}$. Define the function $h(z) = \dfrac{f(z)}{g(z)}$.
Since g is entire, its zeros are isolated. At any point z_0 where $g(z_0) \neq 0$, h is clearly holomorphic.
Suppose g has a zero at z_0 of order m. The condition $|f(z)| \leq |g(z)|$ implies that f must also have a zero at z_0 of order at least m. Consequently, the singularity of h at z_0 is removable. By defining $h(z_0)$ as the limit, h extends to an **entire function**.
From the given inequality, we have:

$$|h(z)| = \left|\frac{f(z)}{g(z)}\right| \leq 1 \quad \forall z \in \mathbb{C}$$

By **Liouville's Theorem**, every bounded entire function is constant. Thus, $h(z) = c$ for some $c \in \mathbb{C}$. Using the initial condition $f(0) = g(0) \neq 0$:

$$c = h(0) = \frac{f(0)}{g(0)} = 1$$

Since $h(z) = 1$ for all $z \in \mathbb{C}$, it follows that $f(z) = g(z)$.

Q 89. Show that a single valued analytic branch of $\sqrt{1-z^2}$ can be defined in any domain such that the points ± 1 are in the same component of the complement.

Solution: A single-valued analytic branch of $f(z) = \sqrt{g(z)}$ exists in a domain Ω if and only if $g(z)$ is non-vanishing in Ω and for every closed curve $\gamma \subset \Omega$, the change in the argument of $g(z)$ along γ is an even multiple of 2π:

$$\Delta_\gamma \arg(1-z^2) = 4\pi k, \quad k \in \mathbb{Z}$$

Equivalently, this requires the winding number of the image path around the origin to be even.
We factor the expression inside the square root:

$$1 - z^2 = (1-z)(1+z) = -(z-1)(z+1)$$

Using the property that the argument of a product is the sum of the arguments, for any closed curve γ in Ω:

$$\Delta_\gamma \arg(1-z^2) = \Delta_\gamma \arg(z-1) + \Delta_\gamma \arg(z+1)$$

In terms of winding numbers $n(\gamma, a)$, this becomes:

$$\Delta_\gamma \arg(1-z^2) = 2\pi n(\gamma, 1) + 2\pi n(\gamma, -1)$$

By hypothesis, the points 1 and -1 are in the same connected component of $\mathbb{C} \setminus \Omega$. A fundamental theorem in topology states that if two points a and b lie in the same component of the complement of a domain Ω, then every closed curve γ in Ω winds around a and b the same number of times:

$$n(\gamma, 1) = n(\gamma, -1)$$

Let $n(\gamma, 1) = n(\gamma, -1) = k$. Substituting this into our argument formula:

$$\Delta_\gamma \arg(1-z^2) = 2\pi k + 2\pi k = 4\pi k$$

We define the branch locally as $f(z) = \exp\left(\dfrac{1}{2}\log(1-z^2)\right)$. After traversing γ, the value of the logarithm increases by $i\Delta_\gamma \arg(1-z^2) = i4\pi k$. The value of $f(z)$ changes by the factor:

$$e^{\frac{1}{2}(i4\pi k)} = e^{i2\pi k} = 1$$

Since the factor is 1, the function returns to its original value, making $f(z)$ a well-defined single-valued analytic function in Ω.

Continued...

Q 90. Prove Gauss's Test: If for sufficiently large n, $\left|\frac{c_{n+1}}{c_n}\right| \leq a + \frac{a}{n}$, where a does not depend on n and $a < -1$, then the series $\sum_{n=1}^{\infty} c_n$ converges absolutely.

Solution: To prove absolute convergence, we examine the series of absolute values $\sum a_n$, where $a_n = |c_n|$. The given condition is:

$$\frac{a_{n+1}}{a_n} \leq a\left(1 + \frac{1}{n}\right)$$

Since $a < -1$, let $p = -a$, where $p > 1$. The inequality becomes:

$$\frac{a_{n+1}}{a_n} \leq -p\left(1 + \frac{1}{n}\right)$$

Gauss's test is a refinement of Raabe's Test. Raabe's Test states that $\sum a_n$ converges if:

$$\lim_{n\to\infty} n\left(1 - \frac{a_{n+1}}{a_n}\right) = L > 1$$

Let's apply this to our given bound:

$$1 - \frac{a_{n+1}}{a_n} \geq 1 - \left(a + \frac{a}{n}\right) = 1 - a - \frac{a}{n}$$

Multiplying by n:

$$n\left(1 - \frac{a_{n+1}}{a_n}\right) \geq n(1-a) - a$$

Since $a < -1$, the term $(1 - a)$ is a positive constant greater than 2. Therefore:

$$\lim_{n\to\infty}[n(1-a) - a] = \infty$$

Because $\infty > 1$, the limit of the Raabe expression is greater than 1. By the Ratio Comparison Test (comparing with a p-series where $p > 1$) or Raabe's Test, the series $\sum |c_n|$ converges. Consequently, the original series $\sum c_n$ converges absolutely.

Q 91. Find the condition on a, b, c under which the equation $az + b\overline{z} + c = 0$ in one complex unknown z has exactly one solution, and compute that solution.

Solution: The original equation is:

$$az + b\overline{z} = -c \tag{1}$$

Taking the complex conjugate of the entire equation:

$$\overline{az} + \overline{b}z = -\overline{c} \implies \overline{b}z + \overline{az} = -\overline{c} \tag{2}$$

To eliminate $\overline{z}$, we multiply equation (1) by $\overline{a}$ and equation (2) by b:

$$\overline{a}(az + b\overline{z}) = \overline{a}(-c) \implies |a|^2 z + \overline{a}b\overline{z} = -\overline{a}c$$
$$b(\overline{b}z + \overline{a}\overline{z}) = b(-\overline{c}) \implies |b|^2 z + b\overline{a}\overline{z} = -b\overline{c}$$

Subtracting the second result from the first:

$$(|a|^2 - |b|^2)z = b\overline{c} - \overline{a}c$$

The equation has exactly one solution if and only if the coefficient of z is non-zero:

$$|a|^2 - |b|^2 \neq 0 \implies |a| \neq |b|$$

Under this condition, the unique solution is:

$$z = \frac{b\overline{c} - \overline{a}c}{|a|^2 - |b|^2}$$

Q 92. Find the most general harmonic polynomial of the form $h = ax^3 + bx^2y + cxy^2 + dy^3$ and find an analytic function f such that $h = \text{Re}(f)$.

Solution: Let $h(x, y) = ax^3 + bx^2y + cxy^2 + dy^3$. For h to be harmonic, it must satisfy the Laplace equation $\Delta h = 0$, where $\Delta h = \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2}$. First, we compute the partial derivatives:

$$\begin{aligned} h_x &= 3ax^2 + 2bxy + cy^2 \\ h_{xx} &= 6ax + 2by \\ h_y &= bx^2 + 2cxy + 3dy^2 \\ h_{yy} &= 2cx + 6dy \end{aligned}$$

Setting the Laplacian to zero:

$$\Delta h = (6ax + 2by) + (2cx + 6dy) = (6a + 2c)x + (2b + 6d)y = 0$$

For this to hold for all x, y, the coefficients must be zero:

$$6a + 2c = 0 \implies c = -3a$$
$$2b + 6d = 0 \implies b = -3d$$

Thus, the most general harmonic polynomial is:

$$h(x, y) = ax^3 - 3dx^2y - 3axy^2 + dy^3$$

We seek an analytic function $f(z)$ such that $\text{Re}(f) = h$. Let $z = x + iy$. Consider the function:

$$f(z) = (a + id)z^3$$

Expanding $f(z)$:

$$\begin{aligned} f(z) &= (a + id)(x + iy)^3 \\ &= (a + id)(x^3 + 3ix^2y - 3xy^2 - iy^3) \\ &= (ax^3 + 3iax^2y - 3axy^2 - iay^3) + (idx^3 - 3dx^2y - 3idxy^2 + dy^3) \\ &= (ax^3 - 3dx^2y - 3axy^2 + dy^3) + i(dx^3 + 3ax^2y - 3dxy^2 - ay^3) \end{aligned}$$

The real part matches our $h(x, y)$. Therefore, the analytic function is:

$$f(z) = (a + id)z^3 + iC \quad \text{where } C \in \mathbb{R}$$

Q 93. Let function $f : \Omega \to \mathbb{C}$ be analytic in a domain $\Omega \subset \mathbb{C}$. Prove that if there exists $c \in \mathbb{C}$ such that $f(z) = c\overline{f(z)}$ for every $z \in \Omega$, then f is constant.

Solution: If $c = 0$, then $f(z) = 0 \cdot \overline{f(z)} = 0$ for all $z \in \Omega$, which is a constant function.
Now, assume $c \neq 0$. If f is identically zero, it is constant. If f is not identically zero, taking the modulus of both sides gives:

$$|f(z)| = |c||\overline{f(z)}| = |c||f(z)|$$

Since $f(z) \neq 0$ for some z, we must have $|c| = 1$. Let $c = e^{i\theta}$ for some $\theta \in \mathbb{R}$. The given equation becomes:

$$f(z) = e^{i\theta}\overline{f(z)}$$

Multiply both sides by $e^{-i\theta/2}$:

$$e^{-i\theta/2}f(z) = e^{i\theta/2}\overline{f(z)}$$

Note that the right side is the complex conjugate of the left side:

$$e^{-i\theta/2}f(z) = \overline{e^{-i\theta/2}f(z)}$$

Let $h(z) = e^{-i\theta/2}f(z)$. Since f is analytic and $e^{-i\theta/2}$ is a constant, h is also analytic on Ω. However, the equation above shows that $h(z)$ is real-valued for all $z \in \Omega$.
By the Cauchy-Riemann equations, any analytic function that maps to a subset of the real line (or any line in the complex plane) must be constant. Thus, $h(z) = k$ for some $k \in \mathbb{R}$. It follows that:

$$f(z) = ke^{i\theta/2}$$

which is constant.

 Continued...

Q 94. Suppose that T is a linear operator such that $T(0) = 1$, $T(1) = i$ and $Y(\infty) = 0$. Find $T(i)$ and describe $T(\mathbb{R})$.

Solution: A linear operator is defined by the form:

$$T(z) = \frac{az + b}{cz + d}$$

Using the given conditions:

- **Condition** $T(\infty) = 0$**:**

$$\lim_{z \to \infty} \frac{az + b}{cz + d} = \frac{a}{c} = 0 \implies a = 0$$

  Thus, $T(z) = \dfrac{b}{cz + d}$.

- **Condition** $T(0) = 1$**:**

$$T(0) = \frac{b}{d} = 1 \implies b = d$$

  The formula simplifies to $T(z) = \dfrac{d}{cz + d} = \dfrac{1}{\frac{c}{d}z + 1}$.

- **Condition** $T(1) = i$**:**

$$T(1) = \frac{1}{\frac{c}{d} + 1} = i \implies \frac{c}{d} + 1 = \frac{1}{i} = -i \implies \frac{c}{d} = -1 - i$$

Substituting these back, we find the general formula:

$$T(z) = \frac{1}{1 - (1 + i)z}$$

Substitute $z = i$ into the formula:

$$T(i) = \frac{1}{1 - (1 + i)i} = \frac{1}{1 - (i + i^2)} = \frac{1}{1 - (i - 1)} = \frac{1}{2 - i} = \frac{2}{5} + \frac{1}{5}i$$

Möbius transformations map lines and circles to lines and circles. Since the real line $\mathbb{R}$ passes through the points $0, 1$, and ∞, their images will define $T(\mathbb{R})$:

- $T(0) = 1$
- $T(1) = i$
- $T(\infty) = 0$

Since the images $\{1, i, 0\}$ are not collinear, $T(\mathbb{R})$ is the unique **circle** passing through these three points. This circle has its center at $\frac{1}{2} + \frac{i}{2}$ and a radius of $R = \frac{1}{\sqrt{2}}$. The set is described by:

$$T(\mathbb{R}) = \left\{ z \in \mathbb{C} : \left| z - \frac{1+i}{2} \right| = \frac{1}{\sqrt{2}} \right\}$$

Q 95. Prove that

(a) the intersection of any number of closed sets in $\mathbb{C}$ is closed.

(b) the union of finite collection of closed sets in $\mathbb{C}$ is closed.

Solution:

(a) Let $\{F_\alpha\}_{\alpha \in I}$ be any arbitrary collection of closed sets in $\mathbb{C}$. And let $F = \bigcap_{\alpha \in I} F_\alpha$. To show that F is closed, we must show that its complement F^c is open.

- Consider the complement of the intersection:

$$F^c = \left(\bigcap_{\alpha \in I} F_\alpha \right)^c$$

- By De Morgan's Laws, the complement of the intersection is the union of the complements:

$$\left(\bigcap_{\alpha \in I} F_\alpha \right)^c = \bigcup_{\alpha \in I} F_\alpha^c$$

- Since each F_α is closed, by definition, each complement F_α^c is an open set.
- A fundamental property of metric spaces (and topology) is that the arbitrary union of open sets is open. Therefore, $\bigcup_{\alpha \in I} F_\alpha^c$ is open.
- Since F^c is open, its complement F is closed.

(b) Let $F_1, F_2, \cdots, F_n$ be a finite collection of closed sets in $\mathbb{C}$. And let $F = \bigcup_{i=1}^{n} F_i$. To show that F is closed, we show that F^c is open.

- Consider the complement of the finite union:

$$F^c = \left(\bigcup_{i=1}^{n} F_i \right)^c$$

- By De Morgan's Laws, the complement of the union is the intersection of the complements:

$$\left(\bigcup_{i=1}^{n} F_i\right)^c = \bigcap_{i=1}^{n} F_i^c$$

- Since each F_i is closed, each complement F_i^c is open.
- A fundamental property of metric spaces is that the finite intersection of open sets is open. Since n is finite, $\bigcap_{i=1}^{n} F_i^c$ is open.
- Since F^c is open, its complement F is closed.

Q 96. Find the center and the radius of the circle which circumscribes the triangle with vertices $0, a_1, a_2 \in \mathbb{C}$.

Solution: Given a triangle with vertices at the origin 0, and two other points a_1 and a_2 in the complex plane, we seek the center c and the radius R of the circumscribing circle. The circumcenter c must satisfy the condition that it is equidistant from all three vertices:

$$|c-0|^2 = |c-a_1|^2 = |c-a_2|^2$$

Expanding $|c|^2 = |c-a_1|^2$ we get:

$$c\bar{c} = (c-a_1)(\bar{c}-\bar{a}_1) \implies c\bar{a}_1 + \bar{c}a_1 = |a_1|^2$$

Similarly, for the second vertex:

$$c\bar{a}_2 + \bar{c}a_2 = |a_2|^2$$

This system of linear equations in c and $\bar{c}$ can be solved using Cramer's rule or substitution. The solution for the center c is:

$$c = \frac{a_1|a_2|^2 - a_2|a_1|^2}{a_1\bar{a}_2 - \bar{a}_1 a_2} = \frac{a_1|a_2|^2 - a_2|a_1|^2}{2i \cdot \mathrm{Im}(\bar{a}_1 a_2)}$$

The radius R is the distance from the circumcenter to any vertex. Taking the distance to the origin:

$$R = |c-0| = |c| = \left|\frac{a_1|a_2|^2 - a_2|a_1|^2}{a_1\bar{a}_2 - \bar{a}_1 a_2}\right|$$

Q 97. Compute $\displaystyle\int_{|z|=1} |z^5 - 1|^2 \, dz.$

Solution: Let us parameterize the countour $|z| = 1$ as:

$$z = e^{i\theta}, \quad \theta \in [0, 2\pi]$$

The differential element of arc length is:

$$|dz| = |ie^{i\theta}|\, d\theta = d\theta$$

Substitute $z = e^{i\theta}$ into the expression $|z^5 - 1|^2$:

$$\begin{aligned}
|e^{i5\theta} - 1|^2 &= (e^{i5\theta} - 1)\overline{(e^{i5\theta} - 1)} \\
&= (e^{i5\theta} - 1)(e^{-i5\theta} - 1) \\
&= e^0 - e^{i5\theta} - e^{-i5\theta} + 1 \\
&= 2 - (e^{i5\theta} + e^{-i5\theta})
\end{aligned}$$

Using the identity $2\cos(nx) = e^{inx} + e^{-inx}$, we get:

$$|z^5 - 1|^2 = 2 - 2\cos(5\theta)$$

Substitute the simplified expression back into the integral:

$$\begin{aligned}
I &= \int_0^{2\pi} (2 - 2\cos(5\theta))\, d\theta \\
&= \left[2\theta - \frac{2}{5}\sin(5\theta)\right]_0^{2\pi} \\
&= \left(2(2\pi) - \frac{2}{5}\sin(10\pi)\right) - \left(0 - \frac{2}{5}\sin(0)\right) \\
&= 4\pi - 0 \\
&= 4\pi
\end{aligned}$$

Q 98. Compute $\displaystyle\int_{|z|=1} |z^5 - 1|^4\, dz$.

Solution: On the unit circle, we have $|z| = 1$, which implies $\overline{z} = \dfrac{1}{z}$. Using the identity $|w|^2 = w \cdot \overline{w}$, we can write:

$$|z^5 - 1|^2 = (z^5 - 1)(\overline{z^5 - 1}) = (z^5 - 1)(\overline{z}^5 - 1)$$

Substituting $\overline{z} = \dfrac{1}{z}$:

$$|z^5 - 1|^2 = (z^5 - 1)\left(\frac{1}{z^5} - 1\right) = (z^5 - 1)\left(\frac{1 - z^5}{z^5}\right) = -\frac{(z^5 - 1)^2}{z^5}$$

Now, we square this expression to find $|z^5 - 1|^4$:

$$|z^5 - 1|^4 = \left(-\frac{(z^5 - 1)^2}{z^5}\right)^2 = \frac{(z^5 - 1)^4}{z^{10}}$$

The integral becomes:

$$I = \int_{|z|=1} \frac{(z^5-1)^4}{z^{10}}\,dz$$

Expanding the numerator using the Binomial Theorem:

$$(z^5-1)^4 = \binom{4}{0}(z^5)^4 - \binom{4}{1}(z^5)^3 + \binom{4}{2}(z^5)^2 - \binom{4}{3}(z^5)^1 + \binom{4}{4}$$

$$(z^5-1)^4 = z^{20} - 4z^{15} + 6z^{10} - 4z^5 + 1$$

The integrand is:

$$f(z) = \frac{z^{20} - 4z^{15} + 6z^{10} - 4z^5 + 1}{z^{10}}$$

Dividing each term by z^{10} gives the Laurent expansion around $z = 0$:

$$f(z) = z^{10} - 4z^5 + 6 - \frac{4}{z^5} + \frac{1}{z^{10}}$$

The Residue Theorem states that the integral is $2\pi i$ times the sum of the residues inside the contour. The residue at $z = 0$ is the coefficient of the $\frac{1}{z}$ term.

- There is no $\frac{1}{z}$ term in the expansion.
- Therefore, $\text{Res}(f, 0) = 0$.

Thus:

$$\int_{|z|=1} |z^5 - 1|^4\,dz = 2\pi i(0) = 0$$

Q 99. Let $f : \mathbb{D} \to \mathbb{C}$ be analytic in $\mathbb{D} = \{z : |z| = 1\}$ is the open unit disc in $\mathbb{C}$. Suppose there exists $\delta > 0$ such that for every real θ with $|\theta| < \delta$, $\lim_{z \to e^{i\theta}} f(z) = 0$. Prove that $f \equiv 0$ on $\mathbb{D}$.

Solution: To prove that $f \equiv 0$, we utilize the **Schwarz Reflection Principle** and the **Identity Theorem**. Let $I = (-\delta, \delta)$ be the open interval on the real axis where f vanishes in the limit. We define a new function $F(z)$ on the domain $G = \mathbb{D} \cup I \cup \mathbb{D}^*$, where $\mathbb{D}^* = \{z \in \mathbb{C} : \overline{z} \in \mathbb{D}\}$ is the lower half-plane:

$$F(z) = \begin{cases} f(z) & z \in H \\ 0 & z \in I \\ \overline{f(\overline{z})} & z \in H^* \end{cases}$$

Using $\overline{f(\overline{z})}$ ensures the extension is analytic. By the **Schwarz Reflection Principle**, since f is analytic in $\mathbb{D}$ and approaches a real-valued constant (zero) continuously on the boundary segment I, the function $F(z)$ is analytic on the union G. Specifically, the continuity of F across the real segment I is guaranteed by the given limit $\lim_{z\to x} f(z) = 0$. We observe the following:

- $F(z)$ is analytic on the connected domain G.
- $F(z) = 0$ for all $z \in I$.
- The interval $I = (-\delta, \delta)$ is a set that contains limit points within G.

According to the **Identity Theorem**, if an analytic function on a connected domain vanishes on a set containing a limit point, it must be identically zero on the entire domain. Thus, $F(z) \equiv 0$ for all $z \in G$.
Therefore, $f \equiv 0$ for all $z \in \mathbb{D}$.

Q 100. If $P(z)$ is a polynomial and C denotes the circle $|z-a| = R$, what is the value of $\int_C \overline{P(z)}\, dz$?

Solution: As $|z-a| = R$, then, $|z-a|^2 = R^2$.
Thus

$$\overline{(z-a)}(z-a) = R^2 \implies \overline{z} = \overline{a} + \frac{R^2}{z-a}$$

The differential $d\overline{z}$ is then:

$$d\overline{z} = -\frac{R^2}{(z-a)^2}\, dz$$

Then,

$$\mathrm{I} = \int_C P(z)\, d\overline{z} = -R^2 \int_C \frac{P(z)}{(z-a)^2}\, dz$$

Using Cauchy Residue Theorem,

$$\begin{aligned}
\mathrm{I} &= -R^2 \cdot 2\pi i \cdot \mathrm{Res}\left(\frac{P(z)}{(z-a)^2}, a\right) \\
&= -R^2 \cdot 2\pi i \cdot \left[\lim_{z\to a} \frac{d}{dz}\left((z-a)^2 \frac{P(z)}{(z-a)^2}\right)\right] \\
&= -R^2 \cdot 2\pi i \cdot \lim_{z\to a} P'(z) \\
&= -2\pi i R^2 \cdot P'(a)
\end{aligned}$$

Therefore,

$$\int_C \overline{P(z)}\, dz = \overline{\int_C P(z)\, d\overline{z}} = \overline{I} = \overline{-2\pi i R^2 \cdot P'(a)} = 2\pi i R^2 \cdot \overline{P'(a)}$$

Q 101. Find the following integral where $\gamma = \{z : |z| = 1\}$.

$$I = \int_\gamma \frac{z + 2z^2 + 3z^3 + 4z^4 + 5z^5}{z^2(2z-1)(3z-2)^2(4z-3)^3}\,dz$$

Solution: To evaluate this integral, we use the **Residue Theorem**. Let the integrand be denoted by $f(z)$:

$$f(z) = \frac{5z^5 + 4z^4 + 3z^3 + 2z^2 + z}{z^2(2z-1)(3z-2)^2(4z-3)^3}$$

The singularities of $f(z)$ occur where the denominator is zero:

1. $z^2 = 0 \implies z = 0$ (Pole of order 2)
2. $2z - 1 = 0 \implies z = \frac{1}{2}$ (Simple pole)
3. $(3z-2)^2 = 0 \implies z = \frac{2}{3}$ (Pole of order 2)
4. $(4z-3)^3 = 0 \implies z = \frac{3}{4}$ (Pole of order 3)

We check if these singularities lie inside the contour $\gamma = \{z : |z| = 1\}$:

- $|0| = 0 < 1$ (Inside)
- $|\frac{1}{2}| = 0.5 < 1$ (Inside)
- $|\frac{2}{3}| \approx 0.67 < 1$ (Inside)
- $|\frac{3}{4}| = 0.75 < 1$ (Inside)

All poles of $f(z)$ lie strictly inside the unit circle. According to the residue theorem, the integral is:

$$I = 2\pi i \sum_k \text{Res}\,(f, z_k)$$

We use the property that for any function $f(z)$ with a finite number of poles in the complex plane:

$$\sum \text{Res}(f, z_k) + \text{Res}\,(f, \infty) = 0 \implies \sum \text{Res}(f, z_k) = -\text{Res}(f, \infty)$$

Since all poles are inside γ, we have:

$$I = 2\pi i\,(-\text{Res}\,(f, \infty))$$

For a rational function $f(z) = \dfrac{P(z)}{Q(z)}$, the residue at infinity depends on the degrees of the polynomials.

- Degree of numerator $P(z)$: $n = 5$.
- Degree of denominator $Q(z)$: Sum the exponents of the factors $\implies 2 + 1 + 2 + 3 = 8$. So, $m = 8$.

A key property of rational functions is that if the degree of the denominator m is at least 2 greater than the degree of the numerator n ($m \geq n + 2$), then:

$$\text{Res}(f, \infty) = 0$$

In our case, $m = 8$ and $n = 5$. Since $8 \geq 5 + 2$ (i.e., $8 \geq 7$), the condition is satisfied. Substituting the residue at infinity back into our integral formula:

$$I = 2\pi i(0) = 0$$

Q 102. Let $f(z)$ be an entire function.

(a) Suppose that $|f(z)| \leq A|z|^n + B$ for all $z \in \mathbb{C}$, where N, A, and B are finite positive constants. Show that f is a polynomial of degree N or less.

(b) Suppose that f satisfies the condition: $f(z_n) \to \infty$ whenever $z_n \to \infty$. Show that f is a polynomial.

Solution:

(a) Let $f(z)$ be an entire function. Since $f(z)$ is an entire function, it has a Taylor series expansion valid for all $z \in \mathbb{C}$:

$$f(z) = \sum_{k=0}^{\infty} a_k z^k$$

The coefficients a_k are given by Cauchy's Estimate. For any $R > 0$, let C_R be the circle $|z| = R$:

$$|a_k| \leq \frac{\max\limits_{|z|=R} |f(z)|}{R^k}$$

Using the given growth condition $|f(z)| \leq A|z|^N + B$, we substitute the maximum possible value of $|f(z)|$ on the circle C_R:

$$|a_k| \leq \frac{AR^N + B}{R^k} = AR^{N-k} + \frac{B}{R^k}$$

Now, consider the behavior as $R \to \infty$ for any integer $k > N$:

$$\lim_{R\to\infty} |a_k| \leq \lim_{R\to\infty} \left(\frac{A}{R^{k-N}} + \frac{B}{R^k} \right)$$

Since $k > N$, the exponent $k - N$ is positive, so $R^{k-N} \to \infty$. Thus, the limit of the right-hand side is 0. This implies $a_k = 0$ for all $k > N$. Therefore, the series for $f(z)$ is finite:

$$f(z) = \sum_{k=0}^{N} a_k z^k$$

which is a polynomial of degree at most N.

(b) An entire function $f(z)$ has an isolated singularity at infinity. To classify this singularity, we examine the function $g(w) = f(\frac{1}{w})$ at $w = 0$. The condition $f(z_n) \to \infty$ as $z_n \to \infty$ implies that:

$$\lim_{w \to 0} g(w) = \lim_{z \to \infty} f(z) = \infty$$

By definition, if the limit of a function at an isolated singularity is ∞, that singularity is a **pole**.
If $g(w)$ has a pole at $w = 0$, its Laurent series expansion has only finitely many terms with negative powers of w:

$$g(w) = \frac{a_N}{w^N} + \cdots + \frac{a_1}{w} + \sum_{n=0}^{\infty} b_n w^n$$

Substituting $w = \frac{1}{z}$ back into the expression, we get the expansion for $f(z)$:

$$f(z) = a_N z^N + \cdots + a_1 z + \sum_{n=0}^{\infty} \frac{b_n}{z^n}$$

Since $f(z)$ is entire, it cannot have any terms with negative powers of z (no singularity at $z = 0$). Thus, $b_n = 0$ for all $n \geq 1$. This leaves:

$$f(z) = a_N z^N + \cdots + a_1 z + b_0$$

which is a polynomial.

Alternative: If $f(z)$ were not a polynomial and not a constant, it would have an **essential singularity** at infinity. By the Great Picard Theorem (or Casorati-Weierstrass), $f(z)$ would take on every complex value (with at most one exception) infinitely often in every neighborhood of infinity, contradicting the fact that $|f(z)| \to \infty$.

Q 103. Evaluate the integral using Cauchy Residue.

$$\int_0^{\pi} \frac{d\theta}{1 + a\cos\theta}, \quad -1 < a < 1$$

Continued...

Solution: The integral to evaluate is:

$$I = \int_0^{\pi} \frac{d\theta}{1 + a\cos\theta}, \quad |a| < 1$$

Since the integrand is even and periodic, we can write:

$$I = \frac{1}{2}\int_0^{2\pi} \frac{d\theta}{1 + a\cos\theta}$$

Let $z = e^{i\theta}$, then $d\theta = \dfrac{dz}{iz}$ and $\cos\theta = \dfrac{1}{2}(z + z^{-1})$. The contour C is the unit circle $|z| = 1$.

$$I = \frac{1}{2}\oint_C \frac{1}{1 + \frac{a}{2}(z + z^{-1})}\frac{dz}{iz} = \frac{1}{i}\oint_C \frac{1}{az^2 + 2z + a}\,dz$$

The poles are the roots of $az^2 + 2z + a = 0$:

$$z = \frac{-2 \pm \sqrt{4 - 4a^2}}{2a} = \frac{-1 \pm \sqrt{1 - a^2}}{a}$$

Let:

$$z_1 = \frac{-1 + \sqrt{1 - a^2}}{a}, \quad z_2 = \frac{-1 - \sqrt{1 - a^2}}{a}$$

Since $|a| < 1$, only z_1 lies inside the unit circle ($|z_1| < 1$). The residue at the simple pole z_1 is:

$$\text{Res}\,(f, z_1) = \lim_{z \to z_1}(z - z_1)\frac{1}{a(z - z_1)(z - z_2)} = \frac{1}{a(z_1 - z_2)}$$

Substituting $z_1 - z_2 = \dfrac{2\sqrt{1 - a^2}}{a}$:

$$\text{Res}\,(f, z_1) = \frac{1}{2\sqrt{1 - a^2}}$$

Therefore,

$$I = \frac{1}{i} \cdot 2\pi i \cdot \text{Res}\,(f, z_1) = 2\pi \cdot \frac{1}{2\sqrt{1 - a^2}} = \frac{\pi}{\sqrt{1 - a^2}}$$

Q 104. Evaluate the integral using Cauchy Residue.

$$\int_0^{2\pi} \frac{d\theta}{1 + a\sin\theta}, \quad -1 < a < 1$$

Continued...

Solution: We need to evaluate the integral:

$$I = \int_0^{2\pi} \frac{d\theta}{1 + a\sin\theta}, \quad |a| < 1$$

Let $z = e^{i\theta}$. Then:

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{z - z^{-1}}{2i}, \quad d\theta = \frac{dz}{iz}$$

As θ goes from 0 to 2π, the contour C is the unit circle $|z| = 1$ in the complex plane. Substituting these into the integral:

$$I = \oint_C \frac{1}{1 + a\left(\frac{z - z^{-1}}{2i}\right)} \frac{dz}{iz} = \oint_C \frac{2}{2iz + az^2 - a} dz = \oint_C \frac{2}{az^2 + 2iz - a} dz$$

The poles of the integrand $f(z) = \dfrac{2}{az^2 + 2iz - a}$ are the roots of the denominator $az^2 + 2iz - a = 0$. Using the quadratic formula:

$$z = \frac{-2i \pm \sqrt{(2i)^2 - 4(a)(-a)}}{2a} = \frac{-2i \pm \sqrt{-4 + 4a^2}}{2a} = \frac{-i \pm i\sqrt{1 - a^2}}{a}$$

The two roots are:

$$z_1 = \frac{i\left(-1 + \sqrt{1 - a^2}\right)}{a}, \quad z_2 = \frac{i\left(-1 - \sqrt{1 - a^2}\right)}{a}$$

Since the product of the roots $z_1 z_2 = \dfrac{-a}{a} = -1$, and it can be shown that for $|a| < 1$, $|z_2| > 1$, it follows that $|z_1| < 1$. Thus, only z_1 lies inside the unit circle C. The residue at the simple pole z_1 is:

$$\text{Res}\,(f, z_1) = \lim_{z \to z_1} (z - z_1) \frac{2}{a(z - z_1)(z - z_2)} = \frac{2}{a(z_1 - z_2)}$$

Calculating $z_1 - z_2$:

$$z_1 - z_2 = \frac{i(-1 + \sqrt{1 - a^2}) - i(-1 - \sqrt{1 - a^2})}{a} = \frac{2i\sqrt{1 - a^2}}{a}$$

Substituting this back:

$$\text{Res}\,(f, z_1) = \frac{2}{a\left(\frac{2i\sqrt{1 - a^2}}{a}\right)} = \frac{1}{i\sqrt{1 - a^2}}$$

By the Cauchy Residue Theorem:

$$I = 2\pi i \sum \text{Res} \; = 2\pi i \left(\frac{1}{i\sqrt{1-a^2}} \right) = \frac{2\pi}{\sqrt{1-a^2}}$$

Q 105. Prove that no matter how small $\epsilon > 0$, for sufficiently large n all the zeroes of the function $f_n(z) = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots + \frac{1}{n!z^n}$ are situated inside the circle $|z| < \epsilon$.

Solution: Let $w = \frac{1}{z}$. Substituting this into the given function, we have:

$$f_n(z) = \sum_{k=0}^{n} \frac{1}{k!z^k} = \sum_{k=0}^{n} \frac{w^k}{k!} = P_n(w)$$

where $P_n(w)$ is the n-th partial sum (Taylor polynomial) of the exponential function e^w. As $n \to \infty$, $P_n(w)$ converges uniformly to e^w on any compact subset of the complex plane. Since the exponential function e^w has no zeros in $\mathbb{C}$, Hurwitz's Theorem (or an application of Rouché's Theorem) implies that for any fixed radius $R > 0$, there exists an N such that for all $n > N$, $P_n(w)$ has no zeros in the disk $|w| \leq R$.
To prove the original statement, let $\epsilon > 0$ be given. We choose $R = \frac{1}{\epsilon}$.
From the reasoning above, for sufficiently large n, all zeros w of $P_n(w)$ must satisfy:

$$|w| > R = \frac{1}{\epsilon}$$

Substituting back $w = \frac{1}{z}$, we find that for the zeros of $f_n(z)$:

$$\left| \frac{1}{z} \right| > \frac{1}{\epsilon} \implies |z| < \epsilon$$

Thus, for sufficiently large n, all zeros of $f_n(z)$ lie inside the circle $|z| < \epsilon$.

Q 106. Let $f_n(z)$, for $n = 1, 2, 3, \ldots$ be analytic on a domain $D \subset \mathbb{C}$ and let $f(z) = \sum_{n=1}^{\infty} f_n(z)$. Show that if the series is uniformly convergent on every compact subset of D, then $f(z)$ is analytic on D.

Solution: Let $S_N(z) = \sum_{n=1}^{N} f_n(z)$ be the N-th partial sum of the series. Since each $f_n(z)$ is analytic on D, $S_N(z)$ is also analytic on D for every $N \in \mathbb{N}$.

By the properties of uniform convergence, if a sequence of continuous functions converges uniformly on a set, the limit function is also continuous. Since $S_N(z) \to f(z)$ uniformly on every compact subset $K \subset D$, and each $S_N(z)$ is continuous, it follows that $f(z)$ is continuous on every compact subset of D. Because continuity is a local property, $f(z)$ is continuous on the entire domain D.
To show that $f(z)$ is analytic, we use Morera's Theorem, which states that a continuous function f in D is analytic if $\oint_\Gamma f(z)\,dz = 0$ for every closed contour Γ in D whose interior also lies in D.
Let Γ be any such closed contour (e.g., the boundary of a triangle or a disk) contained in D. Since Γ is a compact set, the convergence $S_N(z) \to f(z)$ is uniform on Γ.
By Cauchy's Theorem, since $S_N(z)$ is analytic, we have:

$$\oint_\Gamma S_N(z)\,dz = 0 \quad \text{for all } N.$$

Due to the uniform convergence of $S_N(z)$ on Γ, we can interchange the limit and the integral:

$$\oint_\Gamma f(z)\,dz = \oint_\Gamma \lim_{N\to\infty} S_N(z)\,dz = \lim_{N\to\infty} \oint_\Gamma S_N(z)\,dz.$$

Substituting the result of Cauchy's Theorem:

$$\oint_\Gamma f(z)\,dz = \lim_{N\to\infty} (0) = 0.$$

Since $f(z)$ is continuous on D and its integral over any closed contour $\Gamma \subset D$ is zero, by Morera's Theorem, $f(z)$ is analytic on D.

Q 107. How many roots does the equation $z^4 - 6z + 3 = 0$ have in $\{z : 1 < |z| < 2\}$?

Solution: To solve this, we use **Rouché's Theorem**. The theorem states that if $f(z)$ and $g(z)$ are analytic inside and on a simple closed contour C, and $|g(z)| < |f(z)|$ for all $z \in C$, then $f(z)$ and $f(z) + g(z)$ have the same number of zeros inside C. Let $P(z) = z^4 - 6z + 3$. Consider the boundary $|z| = 1$. We split $P(z)$ into:

$$f(z) = -6z, \quad g(z) = z^4 + 3$$

On $|z| = 1$:

$$|f(z)| = |-6z| = 6|z| = 6$$

$$|g(z)| = |z^4 + 3| \leq |z|^4 + 3 = 1^4 + 3 = 4$$

Since $|g(z)| < |f(z)|$ on $|z| = 1$, $P(z)$ has the same number of roots as $f(z) = -6z$ inside $|z| < 1$.

$f(z)$ has **1 root** (at $z = 0$).
Consider the boundary $|z| = 2$. We split $P(z)$ into:

$$f(z) = z^4, \quad g(z) = -6z + 3$$

On $|z| = 2$:

$$|f(z)| = |z|^4 = 2^4 = 16$$

$$|g(z)| = |-6z + 3| \leq 6|z| + 3 = 6(2) + 3 = 15$$

Since $|g(z)| < |f(z)|$ on $|z| = 2$, $P(z)$ has the same number of roots as $f(z) = z^4$ inside $|z| < 2$.
$f(z)$ has **4 roots** (at $z = 0$ with multiplicity 4).
The number of roots in the annulus $1 < |z| < 2$ is:

$$(\text{Roots in } |z| < 2) - (\text{Roots in } |z| \leq 1) = 4 - 1 = 3$$

Thus, the equation has **3 roots** in the region $\{z : 1 < |z| < 2\}$.

Q 108. From the theory of Laurent series expansion, it is known that there are constants a_n such that for $1 < |z| < 4$, we have $\displaystyle\sum_{n=-\infty}^{\infty} a_n z^n = \frac{1}{(z^2 - 5z + 4)}$. Find a_{10} and a_{-10} by the method of your choice.

Solution: To find the coefficients a_{10} and a_{-10} for the function $f(z) = \dfrac{1}{z^2 - 5z + 4}$ in the region $1 < |z| < 4$, we use partial fraction decomposition and expand into geometric series valid for the specified annulus. The function can be written as:

$$f(z) = \frac{1}{3}\left(\frac{1}{z-4} - \frac{1}{z-1}\right)$$

We expand each term based on the constraints of the given region:

- **For $\dfrac{1}{z-4}$:** Since $|z| < 4$, then $|\frac{z}{4}| < 1$:

$$\frac{1}{z-4} = \frac{-1}{4(1 - \frac{z}{4})} = -\frac{1}{4}\sum_{n=0}^{\infty}\left(\frac{z}{4}\right)^n = -\sum_{n=0}^{\infty}\frac{z^n}{4^{n+1}}$$

- **For $\dfrac{1}{z-1}$:** Since $|z| > 1$, then, $|\frac{1}{z}| < 1$:

$$\frac{1}{z-1} = \frac{1}{z(1 - \frac{1}{z})} = \frac{1}{z}\sum_{k=0}^{\infty}\left(\frac{1}{z}\right)^k = \sum_{k=0}^{\infty}\frac{1}{z^{k+1}} = \sum_{n=-\infty}^{-1} z^n$$

Therefore, substituting these back into $f(z)$:

$$f(z) = \frac{1}{3}\left(-\sum_{n=0}^{\infty}\frac{z^n}{4^{n+1}} - \sum_{n=-\infty}^{-1} z^n\right)$$

By comparing the result with the general form $\sum_{n=-\infty}^{\infty} a_n z^n$:

- **For a_{10} ($n = 10$):**
$$a_{10} = \frac{1}{3}\left(-\frac{1}{4^{10+1}}\right) = -\frac{1}{3\cdot 4^{11}}$$
- **For a_{-10} ($n = -10$):**
$$a_{-10} = \frac{1}{3}(-1) = -\frac{1}{3}$$

Q 109. Calculate $\displaystyle\int_{|z|=1}\frac{z^7}{27z^8 - 17z^4 + 2iz^2 - 5z + 2}\,dz.$

Solution: Let the integrand be $f(z) = \dfrac{P(z)}{Q(z)}$, where:

$$P(z) = z^7 \quad \text{and} \quad Q(z) = 27z^8 - 17z^4 + 2iz^2 - 5z + 2$$

To determine how many poles lie inside the unit circle $|z| = 1$, we use **Rouché's Theorem**. Let:

$$g(z) = 27z^8 \quad \text{and} \quad h(z) = -17z^4 + 2iz^2 - 5z + 2$$

On the circle $|z| = 1$, we calculate the magnitudes:

- $|g(z)| = |27z^8| = 27(1)^8 = 27.$
- Using the triangle inequality: $|h(z)| \le |-17z^4| + |2iz^2| + |-5z| + |2| = 17(1)^4 + 2(1)^2 + 5(1) + 2 = 17 + 2 + 5 + 2 = 26.$

Since $|g(z)| > |h(z)|$ for all $|z| = 1$, Rouché's Theorem implies that $Q(z) = g(z)+h(z)$ has the same number of roots inside the unit circle as $g(z)$. Because $27z^8$ has 8 roots (all at $z = 0$), **all 8 poles of $f(z)$ lie inside the unit circle**.
By the Residue Theorem, the integral is $2\pi i$ times the sum of the residues of the poles inside the contour. Since all poles of $f(z)$ are inside $|z| = 1$, we can use the identity:

$$\sum \mathrm{Res}(f, z_k) + \mathrm{Res}(f, \infty) = 0 \implies \sum \mathrm{Res}(f, z_k) = -\mathrm{Res}(f, \infty)$$

Thus, the integral is:

$$I = -2\pi i\,\mathrm{Res}(f, \infty)$$

The formula for the residue at infinity is:

$$\text{Res}(f, \infty) = \text{Res}\left(-\frac{1}{w^2} f\left(\frac{1}{w}\right), 0\right)$$

Then, $f(\frac{1}{w})$:

$$\begin{aligned} f(\frac{1}{w}) &= \frac{(\frac{1}{w})^7}{27(\frac{1}{w})^8 - 17(\frac{1}{w})^4 + 2i(\frac{1}{w})^2 - 5(\frac{1}{w}) + 2} \\ &= \frac{w}{27 - 17w^4 + 2iw^6 - 5w^7 + 2w^8} \end{aligned}$$

Then;

$$\begin{aligned} -\frac{1}{w^2} f(\frac{1}{w}) &= -\frac{1}{w^2} \cdot \frac{w}{27 - 17w^4 + 2iw^6 - 5w^7 + 2w^8} \\ &= -\frac{1}{w(27 - 17w^4 + 2iw^6 - 5w^7 + 2w^8)} \end{aligned}$$

As $w \to 0$, the denominator approaches $w(27)$. The Laurent expansion begins with:

$$-\frac{1}{27w} + \dots$$

The residue at $w = 0$ is the coefficient of $\dfrac{1}{w}$, which is $-\dfrac{1}{27}$. Therefore:

$$\text{Res}\ (f, \infty) = -\frac{1}{27}$$

Substituting the residue back into the integral formula:

$$I = -2\pi i \left(-\frac{1}{27}\right) = \frac{2\pi i}{27}$$

Q 110. Calculate $\displaystyle\int_{|z|=1} \frac{z^6}{17z^8 - 7z^4 + 2iz^2 - 5z + 2}\, dz$.

Solution: We are tasked with calculating the following contour integral along the unit circle $|z| = 1$:

$$I = \oint_{|z|=1} \frac{z^6}{17z^8 - 7z^4 + 2iz^2 - 5z + 2}\, dz$$

More formally, using the Residue Theorem, the integral around a closed contour C enclosing all finite singularities is:

$$\oint_C f(z)\,dz = -2\pi i \cdot \text{Res}(f, \infty)$$

The residue at infinity is defined as:

$$\text{Res}\ (f, \infty) = \text{Res}\left(-\frac{1}{w^2} f\left(\frac{1}{w}\right), 0\right) = -\lim_{z\to\infty} z f(z)$$

Substituting our function $f(z)$:

$$\begin{aligned}
-\text{Res}\ (f, \infty) &= \lim_{z\to\infty} z \cdot \frac{z^6}{17z^8 - 7z^4 + 2iz^2 - 5z + 2} \\
&= \lim_{z\to\infty} \frac{z^7}{17z^8 - 7z^4 + 2iz^2 - 5z + 2} \\
&= \lim_{z\to\infty} \frac{\frac{1}{z}}{17 - \frac{7}{z^4} + \frac{2i}{z^6} - \frac{5}{z^7} + \frac{2}{z^8}} \\
&= \frac{0}{17} = 0
\end{aligned}$$

Since the residue at infinity is 0, the integral is:

$$I = -2\pi i(0) = 0$$

Q 111. Show that $\displaystyle\prod_{n=2}^{\infty}\left(1 - \frac{1}{n^2}\right)$ converges and find its value.

Solution: An infinite product of the form $\prod(1 - a_n)$ with $0 < a_n < 1$ converges if and only if the series $\sum a_n$ converges. In this case, $a_n = \dfrac{1}{n^2}$. The series $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n^2}$ is a p-series with $p = 2 > 1$, which is known to converge. Therefore, the product converges to a non-zero value.
We evaluate the k-th partial product, P_k:

$$\begin{aligned}
P_k = \prod_{n=2}^{k}\left(1 - \frac{1}{n^2}\right) &= \prod_{n=2}^{k}\frac{n^2 - 1}{n^2} = \prod_{n=2}^{k}\frac{(n-1)(n+1)}{n \cdot n} \\
&= \left(\frac{1\cdot 3}{2\cdot 2}\right)\cdot\left(\frac{2\cdot 4}{3\cdot 3}\right)\cdot\left(\frac{3\cdot 5}{4\cdot 4}\right)\cdots\left(\frac{(k-1)(k+1)}{k\cdot k}\right)
\end{aligned}$$

$$= \left(\frac{1 \cdot 2 \cdot 3 \cdots (k-1)}{2 \cdot 3 \cdot 4 \cdots k}\right) \cdot \left(\frac{3 \cdot 4 \cdot 5 \cdots (k+1)}{2 \cdot 3 \cdot 4 \cdots k}\right)$$
$$= \left(\frac{1}{k}\right) \cdot \left(\frac{k+1}{2}\right)$$
$$= \frac{k+1}{2k}$$

Taking the limit as $k \to \infty$:

$$\prod_{n=2}^{\infty} \left(1 - \frac{1}{n^2}\right) = \lim_{k\to\infty} \frac{k+1}{2k} = \lim_{k\to\infty} \left(\frac{1}{2} + \frac{1}{2k}\right) = \frac{1}{2}$$

Q 112. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be the function whose Taylor coefficients $\{a_n\}$ are the Fibonacci numbers: $a_0 = 1$, $a_1 = 1$ and $a_{n+2} = a_n + a_{n+1}$ for $n \geq 0$. Prove that f is analytic in some neighborhood of the origin, and also show that $f(z) = \frac{1}{1 - z - z^2}$.

Solution: Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ where $a_0 = 1$, $a_1 = 1$, and $a_{n+2} = a_{n+1} + a_n$ for $n \geq 0$.
To prove $f(z)$ is analytic in a neighborhood of the origin, we determine its radius of convergence R. Using Binet's Formula, the n-th Fibonacci number is given by:

$$a_n = \frac{\phi^{n+1} - \psi^{n+1}}{\sqrt{5}}$$

where $\phi = \frac{1+\sqrt{5}}{2} \approx 1.618$ and $\psi = \frac{1-\sqrt{5}}{2} \approx -0.618$. The growth rate of the coefficients is governed by ϕ:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \phi$$

By the Ratio Test, the radius of convergence is:

$$R = \frac{1}{\phi} = \frac{\sqrt{5}-1}{2} \approx 0.618$$

Since $R > 0$, the power series converges in the disk $|z| < R$, making $f(z)$ analytic in that neighborhood.
We use the recurrence relation $a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$:

$$f(z) = a_0 + a_1 z + \sum_{n=2}^{\infty} a_n z^n$$

$$zf(z) = a_0 z + a_1 z^2 + \sum_{n=2}^{\infty} a_n z^n$$
$$z^2 f(z) = a_0 z^2 + a_1 z^3 + \sum_{n=2}^{\infty} a_n z^n$$

Therefore,

$$f(z) - zf(z) - z^2 f(z) = a_0 + (a_1 - a_0)z + \sum_{n=2}^{\infty} (a_n - a_{n-1} - a_{n-2}) z^n$$

Substituting $a_0 = 1$, $a_1 = 1$, and $a_n - a_{n-1} - a_{n-2} = 0$ for $n \geq 2$, we get;

$$f(z) - zf(z) - z^2 f(z) = 1 \implies f(z)(1 - z - z^2) = 1$$

Therefore

$$f(z) = \frac{1}{1 - z - z^2} \qquad \text{for} \qquad |z| < \frac{\sqrt{5} - 1}{2}$$

Q 113. Does there exist a function f analytic in $\mathbb{D}$ such that $f(\frac{1}{n} = \frac{(-1)^n}{n}$, where $n = 2, 3, \ldots$? Explain.

Solution: Suppose such an analytic function f exists. The sequence of points $z_n = \dfrac{1}{n}$ lies in $\mathbb{D}$ and has a limit point $0 \in \mathbb{D}$. According to the **Identity Theorem**, if two functions analytic on a domain D agree on a set with a limit point in D, they must be identical throughout D.
We consider two subsequences of z_n:

1. **Even indices:** Let $n = 2k$ for $k \in \mathbb{N}$. Then:
$$f\left(\frac{1}{2k}\right) = \frac{(-1)^{2k}}{2k} = \frac{1}{2k}$$
The function $g(z) = z$ is analytic and agrees with $f(z)$ on the set $S_{even} = \{\frac{1}{2}, \frac{1}{4}, \ldots\}$, which has 0 as a limit point. Thus, by the Identity Theorem, $f(z) = z$ for all $z \in \mathbb{D}$.

2. **Odd indices:** Let $n = 2k + 1$ for $k \in \mathbb{N}$. Then:
$$f\left(\frac{1}{2k+1}\right) = \frac{(-1)^{2k+1}}{2k+1} = -\frac{1}{2k+1}$$
The function $h(z) = -z$ is analytic and agrees with $f(z)$ on the set $S_{odd} = \{\frac{1}{3}, \frac{1}{5}, \ldots\}$, which also has 0 as a limit point. Thus, by the Identity Theorem, $f(z) = -z$ for all $z \in \mathbb{D}$.

Combining these results, we must have $z = -z$ for all $z \in \mathbb{D}$, which implies $2z = 0$. This is only true at $z = 0$ and not on the entire disk. This contradiction shows that no such analytic function f can exist.

Q 114. Map the outside of the unit circle onto the complement of the arc $|z| = 1$, $\text{Im}(z) > 0$ so that the points at ∞ correspond to each other.

Solution: We first map the complement of the arc $|z| = 1$, $\text{Im}(z) > 0$ onto the outside of the unit circle so that the points at ∞ correspond to each other. Then consider its inverse.
Consider the map

$$z_1 = \frac{z+1}{z-1}.$$

This maps the arc $|z| = 1, \text{Im}(z) \geq 0$ to a straight line.
Choose $z = i$ so that

$$z_1(i) = \frac{1+i}{-1+i} = \frac{(1+i)(-1-i)}{2} = -i.$$

Since $z = i$ belongs to the specified arc, it follows that z_1 maps the arc to the ray $x = 0, \text{Im}(z) \leq 0$. This map sends ∞ to 1.
Rotating the domain using

$$z_2 = -iz_1$$

takes the ray to the negative real axis, and the composition maps $\infty \mapsto -i$.
Using the standard branch cut for the square root, we use

$$z_3 = z_2^{1/2}$$

to map the region to the right half-plane. Moreover,

$$\infty \mapsto \sqrt{-i} = \left(e^{-\pi i/2}\right)^{1/2} = e^{-\pi i/4}.$$

Now translate and dilate so that $e^{-\frac{\pi i}{4}}$ is sent to 1. This is achieved via

$$z_4 = \sqrt{2}(z_3 + \frac{\sqrt{2}}{2}i) = \sqrt{2}z_3 + i.$$

Finally, we use the map

$$w = \frac{z_4+1}{z_4-1}$$

to map this to $|w| > 1$. It is clear that $\infty \mapsto \infty$.
The full composition is

$$w = \frac{z_4+1}{z_4-1} = \frac{\sqrt{2}z_3+i+1}{\sqrt{2}z_3+i-1} = \frac{\sqrt{2}z_2^{1/2}+i+1}{\sqrt{2}z_2^{1/2}+i-1}$$

Continued...

$$= \frac{\sqrt{2}(-iz_1)^{1/2} + i + 1}{\sqrt{2}(-iz_1)^{1/2} + i - 1}$$
$$= \frac{\sqrt{2}e^{-\pi i/4}z_1^{1/2} + i + 1}{\sqrt{2}e^{-\pi i/4}z_1^{1/2} + i - 1}$$
$$= \frac{(1-i)z_1^{1/2} + i + 1}{(1-i)z_1^{1/2} + i - 1}$$
$$= \frac{(1-i)\sqrt{\dfrac{z+1}{z-1}} + i + 1}{(1-i)\sqrt{\dfrac{z+1}{z-1}} + i - 1}$$

To verify $w(\infty) = \infty$, define $\tilde{w}(z) = w(\frac{1}{z})$. Then,

$$\tilde{w}(z) = \frac{\sqrt{\dfrac{1+z}{1-z}} + i}{\sqrt{\dfrac{1+z}{1-z}} - i}.$$

Clearly $\tilde{w}(0) = \infty$, so $w(\infty) = \infty$. To find the inverse map $f^{-1}(w)$, we set $w = f(z)$ and solve for z.
Now we find the inverse.
Let $u = \sqrt{\dfrac{z+1}{z-1}}$. The expression for w becomes:

$$w = \frac{(1-i)u + (1+i)}{(1-i)u + (i-1)}$$

Observe that $i - 1 = -(1-i)$ and $1 + i = i(1-i)$ (since $i - i^2 = i + 1$). Substituting these into the equation:

$$w = \frac{(1-i)u + i(1-i)}{(1-i)u - (1-i)} = \frac{u+i}{u-1}$$

Thus;

$$w(u-1) = u + i \implies wu - w = u + i$$
$$\implies wu - u = w + i \quad \implies u(w-1) = w + i$$

Therefore,

$$u = \frac{w+i}{w-1} \implies u^2 = \frac{z+1}{z-1}$$

Then,

$$u^2(z-1) = z+1 \implies u^2 z - u^2 = z+1$$
$$\implies z(u^2-1) = u^2+1 \implies z = \frac{u^2+1}{u^2-1}$$

Substitute $u = \dfrac{w+i}{w-1}$ into the equation for z:

$$z = \frac{\left(\dfrac{w+i}{w-1}\right)^2 + 1}{\left(\dfrac{w+i}{w-1}\right)^2 - 1} = \frac{(w+i)^2 + (w-1)^2}{(w+i)^2 - (w-1)^2}$$

Expand the squares:

$$z = \frac{(w^2+2iw-1) + (w^2-2w+1)}{(w^2+2iw-1) - (w^2-2w+1)} = \frac{2w^2+2iw-2w}{2iw+2w-2} = \frac{w^2+(i-1)w}{(i+1)w-1}$$

The inverse map is:

$$f^{-1}(w) = \frac{w^2+(i-1)w}{(i+1)w-1}$$